MATH 112 SET 4 SOLUTIONS

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2.16) We know from pset 2 solutions that if $0 < x < y$ then $0 < \sqrt{x} < \sqrt{y}$. So $p \in E$ translates to the condition $\sqrt{2} < |p| < \sqrt{3}$, $p \in \mathbb{Q}$. Any point $q \in \mathbb{Q}$ has distance $d(q, \{\pm\sqrt{2}, \pm\sqrt{3}\}) > d > 0$ for some d, since $\pm\sqrt{2}$ and $\pm\sqrt{3}$ are not in $\mathbb{Q}$. So $N_d(q)$is either contained in E or does not meet E, hence E is closed and open in $\mathbb{Q}$. Also note $d(p, q) < 2\sqrt{3}$ if $p, q \in E$.

To show E is not compact consider the open cover $E_n = \{x \mid \sqrt{2} + 1/n < |x| < \sqrt{3}\}$, $n \in \mathbb{N}$, which is open just as E since $\sqrt{2} + 1/n \notin \mathbb{Q}$. Their union covers E since E_n contains any $p \in E$ s. t. $d(p, \sqrt{2}) > 1/n$, though the union of any finite subcover, equal to $E_{n_{max}}$, does not contain the point $\sqrt{2} + 1/(n_{max} + 1)$.

2.22) $\mathbb{Q}^n \subset \mathbb{R}^n$ is countable since it is a finite cartesian product of countable sets, dense since given $\vec{r} = (r_1, \dots, r_n)$ can choose $\vec{q} = (q_1, \dots, q_n)$ s. t. $|r_i - q_i| < \epsilon$. Then $||\vec{r} - \vec{q}|| \leq \sqrt{\sum (r_i - q_i)^2} \leq \sum |r_i - q_i| < n\epsilon$ by argument of pset 2 question 2.11b), so $\mathbb{Q}^n$ is dense since ϵ was arbitrary.

2.23) Let X be a separable metric space and let $\{x_i\}$ be a countable dense subset. Then let $V_{ij} = N_{1/j}(x_i)$ to form a countable collection. Given any point $x \in G$ an open set, let $N_r(x) \subset G$, and choose j, x_i s. t. $1/j < r/2$, x_i and $d(x_i, x) < r/2$. Then we will show $x \in V_{ij} \subset N_r(x) \subset G$. As defined, $x \in V_{ij}$. For any $y \in V_{ij}$ have $d(x, y) \leq d(x, x_i) + d(x_i, y) < r \Rightarrow y \in N_r(x)$.

2.24) We construct a countable base of neighborhoods. Fix $\delta > 0$ and choose a finite sequence $\{x_1, \dots, x_n\}$ such that $d(x_i, x_j) > \delta$ for $i \neq j$, and $\nexists\, x \in X$ s. t. $d(x, x_i) \geq \delta$ for each i. We can choose this sequence point by point until no such x exists: if this did not terminate after choosing finitely many x_i, we would have infinitely many and there would be a limit point where the x_i would accumulate to within δ.

If we now let $\delta = 1/n$ and choose sets $\{x_i^{(n)}\}$ as above for each $n \in \mathbb{N}$, then a countable base is $\{N_{1/n}(x_i^{(n)})\}$ for all n, i. Given $x \in G$, G open, $\exists\, N_r(x) \subset G$. Then choose $1/m < r/2$ and there must exist $x_j^{(m)} \in N_{1/m}(x)$, else x would satisfy $d(x, x(m)_i) > 1/m \,\forall\, i$. Then $N_{1/m}(x_j^{(m)}) \subset N_r(x) \subset G$ so this is a ctbl base.

2.26) We have that X has a countable base $V = \{V_i\}$.

Given an open cover $G = \{G_\alpha\}$ of X, choose a countable subcover $G' = \{G_{\alpha_i}\}$ as follows. For each V_i, pick one G_{α_i} containing V_i if such a G_α exists. Thus at most one G_{α_i} is chosen for any V_i so G' is countable. G' covers X since any point $x \in X$ is contained in some V_i for which some containing G_{α_i} would have been chosen; otherwise there would have been no $V_i \in V$, $G_\alpha \in G$ such that $x \in V_i \subset G_\alpha$ which contradicts that G covers or that V_i is a ctbl base.

Now reindex $G' = \{G_n\}_{n=1}^{\infty}$. (The indices G_{α_i} did not necessarily run over all i since G_{α_i} may have been chosen for some V_i.)

Define $F_n = (G_1 \cup \cdots \cup G_n)^c$ in X. Notice $F_n \supset F_{n+1} \, \forall \, n$. So the intersection of any finite subcollection is equal to $F_{n_{max}}$.

Assume that G' has no finite subcover. This means that the intersection of any finite subcollection, equal to $F_{n_{max}}$, is nonempty, while the intersection $\bigcap_{i=1}^{\infty} F_i = \emptyset$. Now choose $E = \{x_i\}_{i=1}^{\infty}$ such that $x_i \in F_i$ for each i. Then E has a limit point x. However, since each F_n contains x_m for all $m > n$, x is a limit point of each F_n; and since each F_n is closed, $x \in F_n \, \forall \, n \Rightarrow x \in \bigcap_{i=1}^{\infty} F_i$, so this is nonempty $\Rightarrow\Leftarrow$. Thus G' has a finite subcover, which is a finite subcover of G so X is compact.