MATH 112 SET 2 SOLUTIONS

ALEX WALDRON

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N. B. Take "countable" to mean Rudin's "at most countable." Thus a "countable" set is either "finite" or "countably infinite." This way any set is either "countable" or "uncountable." On your writeups you may use either terminology freely.

1) One says $A \subset B$ if $a \in A \Rightarrow a \in B$ for any element a. The null set $\emptyset$ has no elements, so $\emptyset \subset S$ is vacuously true for any set S.

Note. If a statement is not false within what we are considering, here "elements of sets," then our "naive set theory" calls it true. It seems stronger to argue the contrapositive here, but using the contrapositive also requires that "vacuously true" is true.

4) Assume the irrationals $I := \mathbb{R} - \mathbb{Q}$ are countable. Then $\mathbb{Q} \cup I = \mathbb{Q} \cup (\mathbb{R} - \mathbb{Q}) = \mathbb{R}$ is countable $\Rightarrow\Leftarrow$. Therefore I is not countable.

5) Let

$$E_x = \{x + \frac{1}{n} \mid n \in \mathbb{N}\}, x \in \mathbb{R}.$$

Such a set has the unique limit point x: for any $r > 0$ choose $n \in \mathbb{N}$ such that $n > 1/r$ (Archimedean property), then $1/n < r$ by (1.18). Then $N_r(x) \ni (x + 1/n) \in E_x$. Since r was arbitrary, x is a limit point of E_x.

Since E_x can be ordered

$$E_x = \{x + 1/1 > \ldots > x + 1/n > x + 1/(n+1) > x + 1/(n+2) > \ldots\}$$

it is isolated (each point has finite distance to its neighbors).

Then $E_0 \cup E_2 \cup E_4$ has three limit points 0, 2, and 4.

7) First prove $\bar{B} \supset \bigcup \bar{S}_i$ for any collection of subsets $\{S_i\}$ of a metric space. Since each $S_i \subset B$, a limit point of any S_i is also a limit point of B and $S_i' \subset B'$. This establishes (b) and half of (a) as special cases of a countably infinite and finite collection.

To complete (a) we need $\bar{B} \subset \bigcup_{i=1}^n \bar{A}_i$. We are given $B \subset \bigcup_{i=1}^n A_i$, so we will show $B' \subset \bigcup_{i=1}^n A_i'$. Prove the contrapositive: $(B')^c \supset (\bigcup_{i=1}^n A_i')^c$. Let c not a limit point of any A_i, then for each i let $N_{r_i}(c)$ be a neighborhood not meeting A_i (except possibly at c). Since there are finitely many A_i let $r = \min\{r_i\}$. Then $N_r(c) - c$ contains no point of any A_i, hence no points of B, so $c \notin B'$. Therefore a limit point of B is a limit point of some A_i, $B' \subset \bigcup_{i=1}^n A_i'$, so $\bar{B} \subset \bigcup_{i=1}^n \bar{A}_i$.

Finally, we need to exhibit $\{A_i\}$ and B such that $B \not\subset \bigcup A_i$. Let $A_i = (\frac{1}{i}, 1) \subset \mathbb{R}$, $i \in \mathbb{N}$. Then $\bar{B} = [0, 1)$ but $\bigcup \bar{A}_i = (0, 1)$. The argument is very similar to (5).

9) (a)-(c) are clear.

(d) Need $(E^o)^c = \overline{E^c}$. For $(\subset)$ need $E^0 \supset (\overline{E^c})^c$: this is true from (c) because $(\overline{E^c})^c$ is open and contained in E. For $(\supset)$, use $E^o \subset E \Rightarrow (E^o)^c \supset (E^c)$, and $(E^o)^c$ is closed, so it contains $\overline{E^c}$ by (2.27).

(e) No. Let $A = \mathbb{R} - x$ for a single point x. Then $\bar{A} = \mathbb{R}$ so $(\bar{A})^o = \mathbb{R}$ but $A^o = A \neq \mathbb{R}$.
(f) No. Take a set with no interior points.

11) If it is a metric, I will write only the proof of the triangle inequality.

(1) No: violates triangle inequality on, for example $(0,1,3)$: $(1-0)^2 + (3-1)^2 = 5 < 3^2 = 9$.

(2) Yes: Two positive real numbers satisfy $a \leq b$ iff $a^2 \leq b^2$ (if $a, b > 0$ and $a \geq b$ then $0 \leq (a-b)(a+b) = a^2 - b^2$). So it suffices to check

$$\begin{aligned}(d_2(x,y) + d_2(y,z))^2 &= (\sqrt{|x-y|} + \sqrt{|y-z|})^2 \\ &= |x-y| + |y-z| + 2\sqrt{|x-y||y-z|} \\ &\geq |x-y| + |y-z| \\ &\geq |x-z| = d_2(x,z)^2 \text{ (second } \geq \text{ was from triangle inequality).}\end{aligned}$$

(3) No. Violates $d(x,y) = 0$ iff $x = y$: $d_3(a, -a) = 0$.

(4) No. Violates $d(x,y) = 0$ iff $x = y$: $d_4(2a, a) = 0$.

(5) Yes. We will use (1.18), particularly the property that if $a, b > 0$ and $a \leq b$ then $1/a \geq 1/b$. If $y \in [x,z]$ then $|x-y| + |y-z| = |x-z|$ so

$$\begin{aligned}d_5(x,y) + d_5(y,z) &= \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|} \\ &\geq \frac{|x-y|}{1+|x-y|+|y-z|} + \frac{|y-z|}{1+|y-z|+|x-z|} \\ &= \frac{|x-y|+|y-z|}{1+|x-y|+|y-z|} \\ &= \frac{|x-z|}{1+|x-z|} = d_5(x,z).\end{aligned}$$

If $y \notin [x,z]$ then say wlog $|x-y| \geq |x-z|$. Then

$$\begin{aligned}1 + |x-y| &\leq \frac{|x-y|}{|x-z|} + |x-y| \text{ since } |x-z| \leq |x-y| \\ &\leq |x-y|\frac{1+|x-z|}{|x-z|} \\ \iff \frac{|x-z|}{1+|x-z|} &\leq \frac{|x-y|}{1+|x-y|} \\ &\leq \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|} \\ d_5(x,z) &\leq d_5(x,y) + d_5(y,z).\end{aligned}$$

Note. In the next psets I will not require you to directly cite any theorem from chapter 1 except the triangle or C-S inequalities on $\mathbb{C}^n$.