MATH 112 SET 10 SOLUTIONS

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Feel free to email waldron@fas.harvard.edu if anything is unclear. Also, if you think I misunderstood any of your arguments or was too harsh, just email me and we can meet to chat about it/negotiate (this applies for any CA).

7.12) On an interval $[t,T]$, we have uniform convergence so $\lim_{n\to\infty}\int_t^T f_n(x)\,dx = \int_t^T f(x)\,dx$.

Note $|f|\le g$ holds as the limit of $|f_n|\le g$. We now choose $A\in\mathbb{R}_+$ and divide the problem into two improper integrals, above and below A, since any integral under any limit here can be split this way. First consider the integral below A.

Let $I_n(t) := \int_t^A f(x)\,dx$, and $I(t) := \int_t^A f(x)\,dx$.

Let $J_n(t) := I_n(t) - \lim_{t'\to 0} I(t') = \int_t^A f_n(x)\,dx - \int_0^A f(x)\,dx$.

The improper integrals of $I(t)$ and $I_n(t)$, namely $\int_0^A f_n(x)\,dx$ and $\int_0^A f(x)\,dx$, both exist: choose any sequence of points $\{t_n\}$ converging monotonically to 0 from above. Then the improper integral is the sum of the series of proper integrals over the intervals $[t_{n+1},t_n]$. Then this converges by the comparison test with the same series for g.

(*) Also, convergence of the improper integral means that for any ϵ, there exists δ s. t. $t<\delta \Rightarrow \int_0^t g(x)\,dx<\epsilon$. Then $|\int_0^t f_n(x)\,dx| \le \int_0^t |f_n(x)|\,dx \le \int_0^t g(x)\,dx < \epsilon$ for all n. The same δ produces the same statement for $f(x)$.

Solution 1) Given $\epsilon>0$, we need to exhibit an M such that there is a δ such that for all $n>M$ and $t<\delta$, $|J_n(t)|\le\epsilon$. Note $|J_n(t)| \le |\int_0^t f_n(x)\,dx| + |\int_t^A (f_n(x)-f(x))\,dx|$. From statement (*), we can choose δ such that that for $t<\delta$, the first term here is less than ϵ. From uniform convergence we can choose M such that the second term is less than ϵ for $n>M$. So for this M and δ, $J_n(t)<2\epsilon$ for $n>M$ and $t<\delta$ as desired.

Solution 2) We need to exchange the order of limits in

$$\int_0^A f(x)\,dx = \lim_{t\to 0} I(t) = \lim_{t\to 0}\lim_{n\to\infty} I_n(t) = \lim_{n\to\infty}\lim_{t\to 0} I_n(t) = \lim_{n\to\infty}\int_0^A f_n(x)\,dx.$$

Since 0 is a limit point of $(0,A]$, from 7.11 this will hold if $I_n(t)\to I(t)$ uniformly on all of $t\in(0,A]$.

We know $I_n(t)\to I(t)$ pointwise for $t\in(0,A]$ by the very first sentence. Note $I_n(t)$ is continuous, so convergence is uniform on compact intervals; also note that clearly for any $\delta\in[t,A]$, $I_n(t) = \int_t^\delta f_n(x)\,dx + \int_\delta^A f_n(x)\,dx$, and likewise for $I(t)$. It is easy to extend uniformity 0: choose δ from (*) which, for all n and $t\le\delta$, gives $|\int_t^\delta (f_n(x)-f(x))\,dx| \le |\int_t^\delta f_n(x)\,dx| + |\int_t^\delta f(x)\,dx| \le 2\epsilon$. So for this δ, we can choose M to bound the integral from δ to A by ϵ, so the whole integral is bounded by 3ϵ for any t and $n>M$; so this δ, M shows uniform convergence on $(0,A]$.

The integral from A to ∞ can be treated likewise by choosing B large instead of δ small throughout.

7.13) (a) We follow the hint. By 7.23, there is a subsequence $\{f_{n_i}\}$ that converges on $\mathbb{Q}$ to function $f(r)$. Relabel this subsequence simply $\{f_n\}$. This is monotonically increasing on $\mathbb{Q}$ since the limit of monotone functions. To extend this to the reals, for $x \notin \mathbb{Q}$, define $f(x)$ as the left limit of $f(r)$ as $r \to x$, $f(x) := \sup_{r<x} f(r)$, and this also increases monotonically.

It remains to show convergence at irrational points where f is continuous, then where discontinuous.

Let f be continuous at $x \notin \mathbb{Q}$; for arbitrary $\epsilon > 0$, choose $\delta > 0$ such that $|f(x) - f(t)| < \epsilon$ if $|x - t| < \delta$. Choose $r_\pm \in \mathbb{Q}$ satisfying $x - \delta < r_- < x < r_+ < x + \delta$. Now choose $M_\pm$ such that $f_n(r_\pm) \leq \epsilon$ if $n > M_\pm$, by convergence on $\mathbb{Q}$; let M be the greater of $M_\pm$.

From monotonicity, we have $f(x) - f(r_-) \leq f(r_+) - f(r_-) \leq 2\epsilon$. Combine this with the the triangle inequality, for $n > M$:

$$|f_n(x) - f(x)| \leq |f_n(x) - f_n(r_-)| + |f_n(r_-) - f(r_-)| + |f(r_-) - f(r_+)| \leq 4\epsilon.$$

There are countably many points at which f is discontinuous (f is monotone, so each discontinuity must be a finite jump, and no rational point of the image is jumped by more than one discontinuity). So by 7.23 choose a further subsequence that also converges at these countably many discontinuities.