Mathematics 112. Real Analysis, Spring 2006, Solutions

Midterm Exam, Thursday, March 23, 10 - 11:30

Instructor: Robert Strain

Instructions: This is a closed notes, closed book exam. If you use a result from class or from the textbook then please state it clearly. Please write all your answers in the exam book. Good luck!

1. (20pts.) Given any collection $\{G_\alpha\}_{\alpha\in A}$ of open sets, state which of the following are true? Which are false? If true, no proof is required. If false, provide a counter example.

   (a) (5pts.) $\bigcup_{\alpha\in A} G_\alpha$ is open for $A = (1,2) = \{x \in \mathbb{R} : 1 < x < 2\}$.

   (b) (5pts.) $\bigcap_{\alpha\in A} G_\alpha$ is open for $A = (2,3) = \{x \in \mathbb{R} : 2 < x < 3\}$.

   (c) (5pts.) $\bigcap_{\alpha=1}^{N} G_\alpha$ is open for $1 \le N < \infty$.

   (d) (5pts.) $\bigcup_{\alpha=1}^{N} G_\alpha$ is open for $1 \le N < \infty$.

   SOLUTION:

   (a) True.

   (b) False. Take $G_\alpha = (3-\alpha, 5+\alpha) \subset \mathbb{R}$. Then $\bigcap_{\alpha\in A} G_\alpha = [0,7]$.

   (c) True.

   (d) True.

2. (20pts.) Let X be a metric space with metric $d(p,q)$ for $p,q \in X$. Define the following terms. If you use an important term from class, then define it.

   (a) (4pts.) The "neighborhood" of a point $p \in X$.

   (b) (4pts.)A "limit point" p of a set $E \subset X$.

   (c) (4pts.) An "interior point" p of a set $E \subset X$.

   (d) (2pts.) A "closed set".

   (e) (2pts.) An "open set".

   (f) (4pts.) A "compact set".

   Hint: Use symbols as much as possible.

   SOLUTION: See Rudin. Ch. 2.

3. (20pts.) Let X be an infinite set. For $p, q \in X$ define

$$d(p,q) = \begin{cases} 1 & \text{if } p \neq q \\ 0 & \text{if } p = q \end{cases}$$

Prove that this is a metric (in other words, write down the properties of a metric and then prove that $d(p,q)$ satisfies them). Which subsets of the resulting metric space are open? Which are closed? Which are compact? All answers require a proof.

SOLUTION: Let's look at the neighborhoods of X.

$$N_r(p) = \{q \in X : d(p,q) < r\} = \begin{cases} \{p\} & \text{if } 0 < r < 1, \\ X & \text{if } r \geq 1. \end{cases}$$

So every subset of X is both open and closed. Compact sets are the ones that have a finite number of elements.

4. (20pts.) Suppose $\{x_n\}$ is a sequence in a metric space X with metric d. Say

$$\lim_{n\to\infty} x_n = x.$$

Prove that $\{x_1, x_2, \ldots\} \cup \{x\}$ is closed. Is it compact in general? Please briefly explain why (or why not) without a proof.

SOLUTION: We will show that $E = \{x_1, x_2, \ldots\} \cup \{x\}$ is closed. Suppose $y \in E$ is a limit point of E. Then for each $r_n = 2^{-n}$ there exists $y_n \in E$, $y_n \neq y$ such that

$$d(y, y_n) < 2^{-n}.$$

Since $\{y_n\} \subset E$. We know that $\lim_{n\to\infty} y_n = x$. Otherwise this would contradict hypothesis (why?).

By the triangle inequality

$$d(x,y) \leq d(x,y_n) + d(y_n, y).$$

Therefore for any $\epsilon > 0$ we can choose a large N so that $d(x,y) < \epsilon$ $\forall n \geq N$. This means $d(x,y) = 0$ and equivalently $x = y$. So E is closed.

E is compact because every infinite subset has limit point x. See p.40 of Rudin or Exercise 26 from Ch. 2 of your homework or the above.

5. (20pts.) Let $\{x_n\} \subset \mathbb{R}$ be a sequence.

   (a) (5pts.) Suppose $|x_n| \leq 2^{-n}$. Prove $\sum_{n=1}^{\infty} x_n$ converges and $|\sum_{n=1}^{\infty} x_n| \leq 2$.

(b) (10pts.) Define the arithmetic mean of the first n terms by

$$\sigma_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

Suppose $\lim_{n\to\infty} x_n = x$. Prove that $\lim_{n\to\infty} \sigma_n = x$.

(c) (5pts.) Alternatively suppose $x_n = 1 + (-1)^n$. Write down the first 7 terms of this sequence. Prove the following:

i. (1pts.) What is $\liminf_{n\to\infty} x_n =$?

ii. (1pts.) What is $\limsup_{n\to\infty} x_n =$?

iii. (1pts.) Does x_n converge? If so, to what?

iv. (2pts.) Does σ_n converge? If so, to what?

SOLUTION:

(a) Suppose $|x_n| \leq 2^{-n}$. Then

$$\left|\sum_{n=1}^{N} x_n\right| \leq \sum_{n=1}^{N} |x_n| \leq \sum_{n=1}^{N} 2^{-n} \leq \sum_{n=0}^{\infty} 2^{-n} = 2.$$

The last line is just the sum of a geometric series. Therefore the series converges absolutely and is bounded in absolute value by 2.

(b) Define the arithmetic mean of the first n terms by

$$\sigma_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

Suppose $\lim_{n\to\infty} x_n = x$. Prove that $\lim_{n\to\infty} \sigma_n = x$.
Notice that $x = \frac{mx}{m}$. So that

$$\sigma_n - x = \frac{1}{n}\sum_{i=1}^{n}(x_i - x).$$

This is the first step.
For $\epsilon > 0$ we can choose N such that

$$|x - x_n| \leq \frac{\epsilon}{2} \quad \forall n \geq N.$$

Since the series converges, $\exists M$ such that $|x - x_n| \leq M$ for all n.
For $m > N$ split the arithmetic mean as

$$\sigma_m - x = \frac{1}{m}\sum_{i=1}^{N}(x_i - x) + \frac{1}{m}\sum_{i=N+1}^{m}(x_i - x)$$

Now by the triangle inequality

$$
\begin{aligned}
|\sigma_m - x| &= \frac{1}{m}\sum_{i=1}^{N}|x_i - x| + \frac{1}{m}\sum_{i=N+1}^{m}|x_i - x| \\
&\leq \frac{1}{m}\sum_{i=1}^{N} M + \frac{1}{m}\sum_{i=N+1}^{m}\frac{\epsilon}{2} \\
&= \frac{NM}{m} + \frac{m-(N+1)}{m}\frac{\epsilon}{2}
\end{aligned}
$$

Since N and M are fixed, we can choose m large enough so that $\frac{NM}{m} < \frac{\epsilon}{2}$, e.g. we choose $m > \frac{2NM}{\epsilon}$. And for these m, $|\sigma_m - x| < \epsilon$.

(c)
   i. $\liminf_{n\to\infty} x_n = 0$.
   ii. $\limsup_{n\to\infty} x_n = 2$.
   iii. x_n does not converge. Actually, $|x_n - x_{n+1}| = 2$.
   iv. But σ_n converges to 1. Actually, $\sigma_{2m} = 1$ and $\sigma_{2m+1} = \frac{2m}{2m+1}$.

6. (10pts. Extra Credit. Do last if at all.) Let $\{x_n\}$ be a bounded sequence of real numbers. Prove that

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = x$$

if and only if

$$\lim_{n\to\infty} x_n = x.$$

Hint: Use the following equivalent definition of the lim sup and lim inf.

(a) If x_n is bounded above, then $\limsup_{n\to\infty} x_n = x$ if and only if
   i. $\forall\epsilon > 0$ there is an N such that $x_n < x + \epsilon$ whenever $n \geq N$ and
   ii. $\forall\epsilon > 0$ and $\forall M$, there is an $n > M$ such that $x - \epsilon < x_n$.

(b) If x_n is bounded below, then $\liminf_{n\to\infty} x_n = x$ if and only if
   i. $\forall\epsilon > 0$ and $\forall M$, there is an $n > M$ such that $x_n < x + \epsilon$ and
   ii. $\forall\epsilon > 0$ there is an N such that $x - \epsilon < x_n$ whenever $n \geq N$.

Note that the N's, M's and ϵ's in (a) and (b) above may all be different.