Mathematics 112. Real Analysis, Spring 2006

Final Exam, May 9 - May 11, Due by 5pm on May 11

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Instructions: This is an open notes, open book exam. However, while in progress, the only person with whom you are allowed to discuss this exam is the instructor (and you are encouraged to email him with questions regarding the statements of the problems). Results from class, from handouts, from assigned homework problems, or from Rudin's text may be freely used; but please state them clearly. Good luck!

1. Given a function $F : \mathbb{R}^2 \to \mathbb{R}$, find conditions on F which allow you to solve the following equation
$$F(F(x, y), y) = 0$$
for y as a function of x near $(0, 0)$. Assume $F(0, 0) = 0$.

2. Let $\{F_\alpha\}_{\alpha\in\Lambda}$ be a collection of closed and bounded subsets of $\mathbb{R}^n$. Suppose $G \subset \mathbb{R}^n$ is open and
$$\cap_{\alpha\in\Lambda} F_\alpha \subset G$$
   (a) Prove that there exists a finite sub-collection $F_{\alpha_1}, \ldots, F_{\alpha_n}$ such that
$$\cap_{j=1}^{n} F_{\alpha_j} \subset G$$
   Hint: Suppose not!

   (b) Does the statement of (a) remain true if the F_α are all closed but not bounded? If so, why? If not, provide a counter example.

3. The binomial theorem states that for any two real numbers $x, y \in \mathbb{R}$ and for any $n = 1, 2, 3, \ldots$ we have
$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$
where
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$
Recall $n! = n(n-1)\cdots 1$, e.g. $3! = 3(2)1 = 6$. By convention $0! = 1! = 1$. Use Taylor's theorem (Theorem 5.15) to prove the binomial theorem.

   Remark (which is irrelevant to the exam): The factorial function, $f(n) = n!$, can be generalized from integers to all real numbers with the Gamma function (p.192).

4. Suppose f is a real, continuously differentiable function on [0,1] with $f(0) = f(1) = 0$ and

$$\int_0^1 [f(x)]^2 dx = 1.$$

   (a) Prove that

$$\int_0^1 x f(x) f'(x) dx = -\frac{1}{2},$$

   and that

$$\int_0^1 [f'(x)]^2 dx \int_0^1 x^2 [f(x)]^2 dx > \frac{1}{4}.$$

   (b) Prove

$$\frac{1}{4} \int_0^1 \frac{[f(x)]^2}{x^2} dx \leq \int_0^1 [f'(x)]^2 dx.$$

   Hint for (b): Consider that $f' = x^{1/2}(x^{-1/2} f)' + (1/2x) f$.

5. Given a function $f : \mathbb{R} \to \mathbb{R}$ define the upper limit at a point x by

$$\limsup_{y \to x} f(y) = \lim_{\delta \downarrow 0} \sup_{\{y: 0 < |x-y| < \delta\}} f(y)$$

   Similarly define the lower limit as

$$\liminf_{y \to x} f(y) = \lim_{\delta \downarrow 0} \inf_{\{y: 0 < |x-y| < \delta\}} f(y)$$

   (a) Give an $\epsilon - \delta$ definition of the upper limit and the lower limit of a function using the standard definitions of the limit, the supremum and the infimum.

   (b) Prove that the following set is closed for any $\lambda \in \mathbb{R}$

$$\{y \in \mathbb{R} : f(y) \leq \lambda\} \tag{1}$$

   if and only if $\forall x \in \mathbb{R}$ we have

$$\liminf_{y \to x} f(y) \geq f(x).$$

   (c) If, in addition to (1), $\{y \in \mathbb{R} : f(y) \geq \lambda\}$ is also closed for any $\lambda \in \mathbb{R}$, prove that f is continuous.

   Hint: Formulate and prove similar condition to the one in (b) for the lim sup.

6. Let $f_n : [0,1] \to \mathbb{R}$ $(n = 1, 2, \ldots)$ be a sequence of differentiable functions such that each $f_n'(x)$ is continuous. Suppose there exists a constant $M > 0$ such that for all n we have

$$\int_0^1 |f_n'(x)|^2 dx \leq M.$$

Further assume $f_n(0) = 0$ for all n. Prove that there exists a subsequence of the f_n's which converges uniformly on $[0, 1]$.

Hint: You may want to try to apply Theorem 7.25 from Rudin.