Problem Set # 5 Solutions

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1. (# 1, pg. 78) Suppose $\lim_{n\to\infty} s_n = L$. We claim that $\lim_{n\to\infty} |s_n| = |L|$. Indeed, take any $\varepsilon > 0$. Then there exists $N = N(\varepsilon)$ such that for all $n > N$ we have $|s_n - L| < \varepsilon$. Then

$$||s_n| - |L|| \leqslant |s_n - L| < \varepsilon,$$

proving that $\lim_{n\to\infty} |s_n| = |L|$. Notice that $\{|s_n|\}$ may be convergent even if $\{s_n\}$ is divergent, for example, $s_n = (-1)^n$.

2. (# 2, pg. 78) We have

$$\lim_{n\to\infty} (\sqrt{n^2+n} - n) = \lim_{n\to\infty} \frac{(\sqrt{n^2+n} - n)(\sqrt{n^2+n} + n)}{\sqrt{n^2+n} + n} =$$

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+n} + n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}} + 1} = \frac{1}{2}$$

3. (# 3, pg. 78) It is enough to show that $\{s_n\}$ is increasing and bounded above. First, let us prove by induction on n that $s_{n+1} > s_n$. This is obvious for $n = 1$. Suppose that $s_n > s_{n-1}$ and prove that $s_{n+1} > s_n$. We have

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} > \sqrt{2 + \sqrt{s_{n-1}}} = s_n,$$

as required. Next, let us prove that $s_n < 2$ for all n. Again, this is obvious for $n = 1$. Suppose we know that $s_{n-1} < 2$. Then

$$s_n = \sqrt{2 + \sqrt{s_{n-1}}} < \sqrt{2 + \sqrt{2}} < \sqrt{4} = 2.$$

4. (# 4, pg. 78) We have

$$s_{2m+1} = \frac{s_{2m-1} + 1}{2}$$

It is easy to prove by induction that s_{2m-1} increases and $s_{2m-1} < 1$ for all m (see the previous problem). So, s_{2m+1} converges and $L = \lim_{m\to\infty} s_{2m-1}$ must satisfy

$$L = \frac{L+1}{2} \quad \Rightarrow \quad L = 1.$$

Similarly,

$$s_{2m} = \frac{1 + 2s_{2m-2}}{4},$$

and again one proves that s_{2m} increases and $s_{2m} < 1$. Thus, $M = \lim_{m\to\infty} s_{2m}$ exists and satisfies

$$M = \frac{1+2M}{4} \quad \Rightarrow \quad M = 1/2.$$

It follows that $\limsup_{n\to\infty} s_n = 1$ and $\liminf_{n\to\infty} s_n = 1/2$.

5. (# 5, pg. 78) If one of the the limits is ∞ the other can only be a finite number or ∞. In both cases $\limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n = \infty$, and there is nothing to prove. If one of the limits if $-\infty$, the other can only be a finite number or $-\infty$, so in both cases $\limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n = -\infty$. On the other hand, if, say $\limsup_{n\to\infty} a_n = -\infty$, then in fact $\lim_{n\to\infty} a_n = -\infty$. On the other hand, since $\limsup_{n\to\infty} b_n < \infty$, we have that $\{b_n\}$ is bounded above, and therefore $\lim_{n\to\infty}(a_n + b_n) = -\infty$, so the required inequality (in fact, equality) holds.

It remains to consider the case where both $L = \limsup_{n\to\infty} a_n$ and $M = \limsup_{n\to\infty} b_n$ are finite. Take any $\varepsilon > 0$. Then for all sufficiently large n we have $a_n < L + \varepsilon$ and $b_n < M + \varepsilon$, and therefore $a_n + b_n < L + M + 2\varepsilon$. It follows that $\limsup_{n\to\infty}(a_n + b_n) \leqslant L + M + 2\varepsilon$. Since this is true for all $\varepsilon > 0$, we get $\limsup_{n\to\infty}(a_n + b_n) \leqslant L + M$, as required.

6. (# 7, pg. 78) It is enough to show that the sequence of partial sums of the series is bounded. Using the Cauchy-Schwarz inequality, we get

$$\sum_{n=1}^{N} \frac{\sqrt{a_n}}{n} \leqslant \left(\sum_{n=1}^{N} (\sqrt{a_n})^2\right)^{1/2} \left(\sum_{n=1}^{N} \frac{1}{n^2}\right)^{1/2}$$

Since the series $\sum a_n$ and $\sum 1/n^2$ converge, their partial sums are bounded, so it follows that the partial sums of the series $\sum \sqrt{a_n}/n$ are also bounded.