

1. (Ch. 2, # 6) We need to show that every limit point of E' belongs to E' . Let p be a limit point of E' . This means that every neighborhood $N_r(p)$ contains a point $q \in E'$, $q \neq p$. Let $s = \min(r - d(q, p), d(q, p)/2)$; then it follows from the triangle inequality that $N_s(q) \subset N_r(p)$ (indeed, if $d(x, q) < s$ then $d(x, p) \leq d(x, q) + d(q, p) < s + d(q, p) = r$). Since $q \in E'$, there exists $q_0 \in N_s(q) \cap E'$. Then $q_0 \in N_r(p) \cap E'$. At the same time $q_0 = p$ is impossible because $d(q_0, q) < s \leq d(p, q)/2 < d(p, q)$. So, $p \in E'$, as required.

Clearly, every limit point of E is also a limit point of $\bar{E} = E \cup E'$, so what we need to show is that every limit point of \bar{E} is a limit point of E . Suppose p is a limit point of \bar{E} which is not a limit point of E . This means that there exists a neighborhood $N_r(p)$ that does not contain any points $q \in E$, $q \neq p$. But $N_s(p)$ for any $s < r$ must contain a point $q \in \bar{E}$, $q \neq p$; then clearly $q \in E'$. It follows that p is a limit point of E' , and then by the first part p is also a limit point of E .

Example: Let $E = \{1/2, 1/3, 1/4, \dots, 1/n, \dots\}$. Then $E' = \{0\}$. But E' does not have any limit points. So, E and E' do not necessarily have the same limit points.

2. (Ch. 2, # 8). Yes, every point of an open set $E \subset \mathbb{R}^2$ is a limit point of E . Indeed, let $p \in E$. Since E is open, there exists $r > 0$ such that $N_r(p) \subset E$. Let $N_s(p)$ be any neighborhood of p . Set $t = \min(r, s)$ and take any $q \in N_t(p)$, $q \neq p$. Then $q \in N_r(p) \subset E$, so $q \in N_s(p) \cap E$. Thus, p is a limit point of E .

Example: Let $E = \mathbb{Z} \times \mathbb{Z}$. Then E is closed in \mathbb{R}^2 , but E has no limit points.

3. (Ch. 2, # 10) Properties (a) and (b) in Definition 2.15 (of a metric) are obvious, so let us property (c). If $p = q$ then trivially

$$d(p, q) = 0 \leq d(p, r) + d(r, q).$$

Suppose $p \neq q$. Then either $p \neq r$ or $r \neq q$. In either case,

$$d(p, q) = 1 \leq d(p, r) + d(r, q).$$

Thus, d is a metric. For any $p \in X$, we have $N_{1/2}(p) = \{p\}$ which implies that every subset is open. Therefore, every subset is also closed. Let $K \subset X$ be compact. Then

$$K \subset \bigcup_{p \in K} N_{1/2}(p)$$

is an open cover of K . Choose a finite subcover $N_{1/2}(p_1), \dots, N_{1/2}(p_r)$. Since $N_{1/2}(p) = \{p\}$ for any p , we actually have $K = \{p_1, \dots, p_r\}$. So, every compact set must be finite. Conversely, every finite set is obviously compact.

4. (Ch. 2, # 12). Let

$$K \subset \bigcup_{\alpha} G_{\alpha}$$

be an open cover of K . There exist α_0 such that $0 \in G_{\alpha_0}$. Then there exists N such that $1/n \in G_{\alpha_0}$ for all $n > N$. For each $k = 1, \dots, N$ find α_k so that $1/k \in G_{\alpha_k}$. Then $G_{\alpha_0}, G_{\alpha_1}, \dots, G_{\alpha_N}$ is the required finite subcover of K .

5. (Ch. 2, # 15) Example 1: Let $K_n = [n, \infty)$, where $n = 1, 2, \dots$; clearly, all K_n are closed. Then

$$\bigcap_{\ell=1}^d K_{n_{\ell}} = K_n \quad \text{where } n = \max(n_1, \dots, n_d),$$

hence nonempty. However,

$$\bigcap_{n=1}^{\infty} K_n = \emptyset.$$

(Notice that these sets satisfy $K_n \supset K_{n+1}$ for all n .)

Example 2: Let $K_n = (0, 1/n)$. Clearly, each K_n is bounded, and we also have the inclusion $K_n \supset K_{n+1}$ for all n . Again, we have

$$\bigcap_{\ell=1}^d K_{n_{\ell}} = K_n \quad \text{where } n = \max(n_1, \dots, n_d),$$

hence nonempty. However,

$$\bigcap_{n=1}^{\infty} K_n = \emptyset.$$

6. (Ch. 2, # 17). E is uncountable because there is a bijection between E and the set of all sequences whose elements are the digits 0 and 1, which is uncountable (Theorem 2.14).

E is not dense in $[0, 1]$ because the neighborhood of zero of radius $r = 0.1$ does not contain any elements of E .

E is compact. Indeed, since it is bounded, we only need to show that E is closed. Let $p = 0.a_1a_2\dots$ be a limit point of E . Take $r = 10^{-(n+1)}$. Then $N_r(p)$ contain a point $q = 0.b_1b_2\dots \in E$. Then $a_k = b_k \in \{4, 7\}$ for all $k = 1, \dots, n$. Since this is true for all n , we get that $a_k \in \{4, 7\}$ for all k , and hence $p \in E$.

E is perfect. Indeed given $p = 0.a_1a_2\dots \in E$ and any neighborhood $N_r(p)$, we can pick n so that $10^{-n} < r$. Take $\bar{a}_{n+1} \in \{4, 7\} \setminus \{a_{n+1}\}$. Then the number q whose decimal expansion is obtained from that of p by replacing a_{n+1} with \bar{a}_{n+1} is different from p and belongs to $N_r(p) \cap E$. So, E does not have isolated points.