Problem Set # 1 Solutions

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1. If $r + x \in \mathbb{Q}$ then $x = (-r) + (r + x) \in \mathbb{Q}$ - a contradiction. Similarly, if $rx \in Q$ then $x = r^{-1}(rx) \in \mathbb{Q}$ - a contradiction.

2. Suppose $q \in Q$ is such that $q^2 = 12$. Write q in the lowest terms:

$$q = \frac{m}{n} \quad \text{where} \quad \text{g.c.d.}(m, n) = 1.$$

Then $m^2 = 12n^2$. In particular, m^2 is a multiple of 3. It follows from unique factorization in $\mathbb{Z}$ that then m must also be a multiple of 3, say $m = 3\ell$. We have $3\ell^2 = 4n^2$. Thus, $4n^2$ is a multiple of 3, and again unique factorization in $\mathbb{Z}$ implies that n is a multiple of 3. But this contradicts our assumption that g.c.d.$(m, n) = 1$.

3. **Proof of Proposition 1.15.** (a)

$$xy = xz \quad \Rightarrow \quad x(y - z) = 0 \quad \Rightarrow x^{-1}(x(y - z)) = x^{-1}0 = 0$$

(see Proposition 1.16(a)). Thus, $y - z = 0$, i.e. $y = z$.
(b) $xy = x = x \cdot 1$ by (a) implies $y = 1$.
(c) $xy = 1 = xx^{-1}$ by (a) implies $y = x^{-1}$.
(d) We have $x^{-1}(x^{-1})^{-1} = 1 = x^{-1}x$, so by (a), $(x^{-1})^{-1} = x$.

4. By Definition 1.7, since α is a lower bound of E, $\alpha \leqslant x$ for all $x \in E$. Similarly, since β is an upper bound of E, $x \leqslant \beta$ for all $x \in E$. Pick $x \in E$. Then we have $\alpha \leqslant x \leqslant \beta$. By transitivity, we conclude that $\alpha \leqslant \beta$.

5. We will prove that $\inf A \leqslant -\sup(-A)$ and $\inf A \geqslant -\sup(-A)$ to conclude that $\inf A = -\sup(-A)$. To prove the first inequality, take any $x \in A$. Then by Definition 1.8, $x \geqslant \inf A$. It follows that $-\inf A \geqslant -x$, which implies that $-\inf A \geqslant y$ for all $y \in -A$. Thus, $-\inf A \geqslant \sup(-A)$, and consequently, $\inf A \leqslant -\sup(-A)$. To prove the second inequality, take any $y \in -A$. Then $y \leqslant \sup(-A)$, which implies that $-y \geqslant -\sup(-A)$. Thus $x \geqslant -\sup(-A)$ for all $x \in A$. Therefore, $\inf A \geqslant -\sup(-A)$. Our claim follows from the two inequalities.

13. Using the Triangle Inequality, we obtain

$$|x| = |(x - y) + y| \leqslant |x - y| + |y| \quad \Rightarrow \quad |x| - |y| \leqslant |x - y|$$

Similarly, one shows that

$$|y| - |x| \leqslant |x - y|$$

Since $|a| = \max(a, -a)$ for any $a \in \mathbb{R}$, we obtain that

$$||x| - |y|| \leqslant |x - y|$$

15. For $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$, let

$$\mathbf{a} \cdot \mathbf{b} = a_1\bar{b}_1 + \cdots + a_n\bar{b}_n$$

be the usual (hermitian) dot-product. Then the Schwartz inequality states that

$$|\mathbf{a} \cdot \mathbf{b}| \leqslant |\mathbf{a}||\mathbf{b}| \tag{1}$$

where $|\mathbf{a}| = (\mathbf{a} \cdot \mathbf{a})^{1/2}$. Suppose the equality holds. One possibility is that $\mathbf{a} = 0$ or/and $\mathbf{b} = 0$. Assume that $\mathbf{a} \neq 0$; then $\mathbf{a} \cdot \mathbf{a} > 0$. Set

$$\lambda = \frac{\mathbf{b} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}.$$

Then

$$(\lambda\mathbf{a} - \mathbf{b}) \cdot (\lambda\mathbf{a} - \mathbf{b}) = \lambda\bar{\lambda} \cdot (\mathbf{a} \cdot \mathbf{a}) - \lambda \cdot (\mathbf{a} \cdot \mathbf{b}) - \bar{\lambda} \cdot (\mathbf{b} \cdot \mathbf{a}) + (\mathbf{b} \cdot \mathbf{b}) =$$

$$= \bar{\lambda}(\lambda \cdot (\mathbf{a} \cdot \mathbf{a}) - (\mathbf{b} \cdot \mathbf{a})) + ((\mathbf{b} \cdot \mathbf{b}) - \lambda \cdot (\mathbf{a} \cdot \mathbf{b})) =$$

$$= \frac{(\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})(\mathbf{b} \cdot \mathbf{a})}{\mathbf{a} \cdot \mathbf{a}} = \frac{|\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a} \cdot \mathbf{b}|^2}{\mathbf{a} \cdot \mathbf{a}} = 0$$

because by our assumption we have equality in (1). It follows that $\mathbf{b} = \lambda\mathbf{a}$. Conversely, if $\mathbf{b} = \lambda\mathbf{a}$ then

$$|\mathbf{a} \cdot \mathbf{b}| = |\lambda||\mathbf{a}|^2 = |\mathbf{a}||\mathbf{b}|,$$

i.e. we have equality in (1). Thus, the equality in (1) is equivalent to the vectors a and b being proportional.

17. We have

$$|\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2 = (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) + (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) =$$

$$\mathbf{x} \cdot \mathbf{x} + 2(\mathbf{x} \cdot \mathbf{y}) + \mathbf{y} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{x} - 2(\mathbf{x} \cdot \mathbf{y}) + \mathbf{y} \cdot \mathbf{y} =$$

$$2(\mathbf{x} \cdot \mathbf{x}) + 2(\mathbf{y} \cdot \mathbf{y}) = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$$

If we consider the parallelogram with the sides $\mathbf{x}$ and $\mathbf{y}$, then the vectors $\mathbf{x} + \mathbf{y}$ and $\mathbf{x} - \mathbf{y}$ represent the diagonals of this parallelogram. So, the equality we proved states that the sum of the squares of (the lengths of) the diagonals equals the perimeter of our parallelogram.