CHAPTER 3

Developments and Applications of the Differential Calculus

3.1 Implicit Functions

a. General Remarks

Frequently in analytical geometry the equation of a curve is given not in the form $y = f(x)$ but in the form $F(x, y) = 0$. A straight line may be represented in this way by the equation $ax + by + c = 0$, and an ellipse, by the equation $x^2/a^2 + y^2/b^2 = 1$. To obtain the equation of the curve in the form $y = f(x)$ we must "solve" the equation $F(x, y) = 0$ for y. In Volume I we considered the special problem of finding the inverse of a function $y = f(x)$, that is, the problem of solving the equation $F(x, y) = y - f(x) = 0$ for the variable x.

These examples suggest the importance of methods for solving an equation $F(x, y) = 0$ for x or for y. We shall find such methods even for equations involving functions of more than two variables.

In the simplest cases, such as the foregoing equations for the straight line and ellipse, the solution can readily be found in terms of elementary functions. In other cases, the solution can be approximated as closely as we desire. For many purposes, however, it is preferable not to work with the solved form of the equation or with these approximations but instead to draw conclusions about the solution by directly studying the function $F(x, y)$, in which neither of the variables x, y is given preference over the other.

Not every equation $F(x, y) = 0$ is the implicit representation of a function $y = f(x)$ or $x = \phi(y)$. It is easy to give examples of equations $F(x, y) = 0$ that permit no solution in terms of functions

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of one variable. Thus, the equation $x^2 + y^2 = 0$ is satisfied by the single pair of values $x = 0$, $y = 0$ only, while the equation $x^2 + y^2 + 1 = 0$ is satisfied by no real values at all. It is therefore necessary to investigate more closely the circumstances under which an equation $F(x, y) = 0$ defines a function $y = f(x)$ and the properties of this function.

Exercises 3.1a

1. Suppose that for some pair of values (a, b), $f(a, b) = 0$. If a is known, give a constructive iterative method for finding b. Under what conditions on f will this method work?

b. Geometrical Interpretation

To clarify the situation we represent the function $F(x, y)$ by the surface $z = F(x, y)$ in three-dimensional space. The solutions of the equation $F(x, y) = 0$ are the same as the simultaneous solutions of the two equations $z = F(x, y)$ and $z = 0$. Geometrically, our problem is to find whether the surface $z = F(x, y)$ intersects the x, y-plane in curves $y = f(x)$ or $x = \phi(y)$. (How *far* such a curve of intersection may extend does not concern us here.)

A first possibility is that the surface and the plane have no point in common. For example the paraboloid $z = F(x, y) = x^2 + y^2 + 1$ lies entirely above the x, y-plane. Here there is no curve of intersection. Obviously, we need consider only cases in which there is at least one point (x_0, y_0) at which $F(x_0, y_0) = 0$; the point (x_0, y_0) constitutes an "initial point" for our solution.

Knowing an initial solution, we have two possibilities: either the tangent plane at the point (x_0, y_0) is horizontal or it is not. If the tangent plane is horizontal, we can readily show by means of examples that it may be impossible to extend a solution $y = f(x)$ or $x = \phi(y)$ from (x_0, y_0). For example, the paraboloid $z = x^2 + y^2$ has the initial solution $x = 0$, $y = 0$, but contains no other point in the x, y-plane. In contrast, the surface $z = xy$ with the initial solution $x = 0, y = 0$ intersects the x, y-plane along the lines $x = 0$ and $y = 0$; but in no neighborhood of the origin can we represent the *whole* intersection by a function $y = f(x)$ or by a function $x = \phi(y)$, (see Figs. 3.1 and 3.2). On the other hand, it is quite possible for the equation $F(x, y) = 0$ to have such a solution even when the tangent plane at the initial solution is horizontal, as in the case $F(x,y) = (y - x)^4 = 0$. In the exceptional case of a horizontal tangent plane, therefore, no definite general statement can be made.

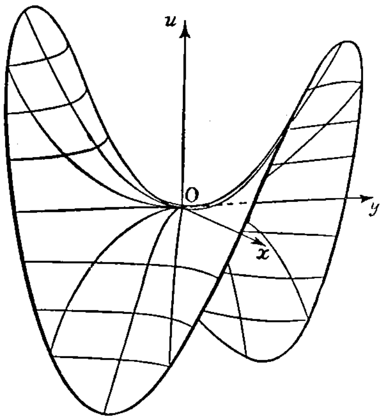


Figure 3.1 The surface $u = xy$.

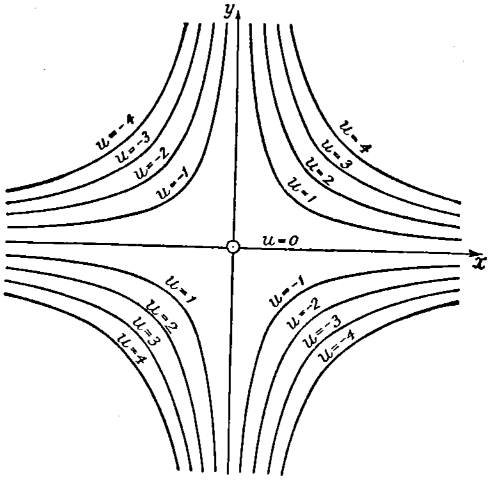
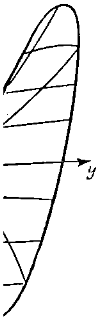


Figure 3.2 Contour lines of $u = xy$.

The remaining possiblity is that the tangent plane at the initial solution is not horizontal. Then, thinking intuitively of the surface $z = F(x, y)$ as approximated by the tangent plane in a neighborhood of the initial solution, we may expect that the surface cannot bend fast enough to avoid cutting the x,y-plane near (x_0, y_0) in a single well-defined curve of intersection and that a portion of the curve near the initial solution can be represented by the equation $y = f(x)$



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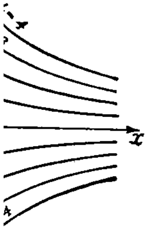

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or $x = \phi(y)$. Analytically, the statement that the tangent plane is not horizontal means that $F_x(x_0, y_0)$ and $F_y(x_0, y_0)$ are not both zero (see p. 47). This is the basis for the discussion in the next subsection.

Exercises 3.1b

1. By examining the surface of $z = f(x, y)$, determine whether the equation $f(x, y) = 0$ can be solved for y as a function of x in a neighborhood of the indicated point (x_0, y_0) for
   (a) $f(x, y) = x^2 - y^2, \quad x_0 = y_0 = 0$
   (b) $f(x, y) = [\log (x + y)]^{1/2}, \quad x_0 = 1.5, \quad y_0 = -.5$
   (c) $f(x, y) = \sin [\pi (x + y)] - 1, \quad x_0 = y_0 = 1/4$
   (d) $f(x, y) = x^2 + y^2 - y, \quad x_0 = y_0 = 0.$

c. *The Implicit Function Theorem*

We now state sufficient conditions for the existence of implicit functions and at the same time give a rule for differentiating them:

Let $F(x, y)$ have continuous derivatives F_x and F_y in a neighborhood of a point (x_0, y_0), where

(1) $$F(x_0, y_0) = 0, \quad F_y(x_0, y_0) \neq 0.$$

Then centered at the point (x_0, y_0), there is some rectangle

(2) $$x_0 - \alpha \leqq x \leqq x_0 + \alpha, \quad y_0 - \beta \leqq y \leqq y_0 + \beta$$

such that for every x in the interval I given by $x_0 - \alpha \leqq x \leqq x_0 + \alpha$ the equation $F(x, y) = 0$ has exactly one solution $y = f(x)$ lying in the interval $y_0 - \beta \leqq y \leqq y_0 + \beta$. This function f satisfies the initial condition $y_0 = f(x_0)$ and, for every x in I,

(3) $$F(x, f(x)) = 0.$$

(3a) $$y_0 - \beta \leqq f(x) \leqq y_0 + \beta$$

(3b) $$F_y(x, f(x)) \neq 0.$$

Furthermore, f is continuous and has a continuous derivative in I, given by the equation

(4) $$y' = f'(x) = -\frac{F_x}{F_y}.$$

This is a strictly *local* existence theorem for solutions of the equation $F(x, y) = 0$ in the neighborhood of an initial solution (x_0, y_0). It does not indicate how to find such an initial solution or how to decide if the equation $F(x, y) = 0$ is satisfied for any (x, y) at all. These are *global* questions and beyond the scope of the theorem. *Uniqueness* and *regularity* of the solution $y = f(x)$, also, can be guaranteed only locally, that is, when y is restricted to the interval $y_0 - \beta < y < y_0 + \beta$. The need for such restrictions is evident from the simple example of the equation

$$F(x, y) = x^2 + y^2 - 1 = 0.$$

For every x with $-1 < x < 1$ the equation has two different solutions $y = \pm\sqrt{1 - x^2}$. A single-valued solution $y = f(x)$ is obtained by prescribing arbitrarily one of the signs at each x. It is clear that in this way we can find solutions that are discontinuous for every x, choosing, for example, the positive sign for rational x and the negative one for irrational x. Continuous solutions $y = f(x)$ are obtained if we restrict y to a constant sign. This sign can be fixed by choosing for a given x_0 in $-1 < x_0 < 1$ one of the two possible values y_0 for which $x_0^2 + y_0^2 = 1$. A unique continuous solution $y = f(x)$ with $y_0 = f(x_0)$ is obtained then for all x in $-1 < x < 1$ by requiring y to satisfy $x^2 + y^2 = 1$ and to have the same sign as y_0. Geometrically, the graph of f is either the upper or the lower semicircle, whichever contains the point (x_0, y_0). The function f has a continuous derivative

$$y' = -\frac{F_x}{F_y} = -\frac{x}{y} = -\frac{x}{f(x)}$$

for $-1 < x < 1$. With y defined to be zero for $x = \pm 1$, the solution $y = f(x)$ will be continuous in the closed interval $-1 \leqq x \leqq 1$. However, the derivative y' then becomes infinite at the end points of the interval, since $F_y = 0$ there.

We shall prove the general theorem in the next section. We observe here only that once the existence and the differentiability of the function $f(x)$ satisfying (3) have been established, we can find an explicit expression for $f'(x)$ by applying the chain rule [see (18) p. 55] to differentiate $F(x, y)$. This yields

$$F_x + F_y f'(x) = 0,$$

and leads to formula (4) as long as $F_y \neq 0$. Equivalently, if the equation $F(x, y) = 0$ determines y as a function of x, we conclude that

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$$dF = F_x\,dx + F_y\,dy = 0$$

and, hence, that

$$dy = \frac{dy}{dx}\,dx = -\frac{F_x}{F_y}\,dx.$$

An implicit function $y = f(x)$ can be differentiated to any given order, provided the function $F(x, y)$ possesses continuous partial derivatives of that same order. For example, if $F(x, y)$ has continuous first and second derivatives in the rectangle (2), the right side of equation (4) is a compound function of x:

$$-\frac{F_x(x, f(x))}{F_y(x, f(x))}.$$

Since, by (3b), the denominator does not vanish and since $f(x)$ already is known to have a continuous first derivative, we conclude from (4) that y' has a continuous derivative; by the chain rule y'' is given by

$$y'' = -\frac{F_yF_{xx} + F_yF_{xy}f' - F_xF_{xy} - F_xF_{yy}f'}{F_y{}^2}.$$

Substituting the expression (4) for f', we find that

$$y'' = -\frac{F_y{}^2F_{xx} - 2F_xF_yF_{xy} + F_x{}^2F_{yy}}{F_y{}^3}. \tag{5}$$

The rules (4) and (5) for finding the derivatives of an implicit function $y = f(x)$ can be used whenever the existence of f in an interval has been established from the general theorem on implicit functions, even in cases where it is impossible to express y explicitly in terms of elementary functions (rational functions, trigonometric functions, etc.). Even if we can solve the equation $F(x, y) = 0$ explicitly for y, it is usually easier to find the derivatives of y from the formulae (4) and (5), without making use of any explicit representation of $y = f(x)$.

Examples

1. The equation of the *lemniscate* (Volume I, p. 102)

$$F(x, y) = (x^2 + y^2)^2 - 2a^2(x^2 - y^2) = 0$$

is not easily solved for y. For $x = 0$, $y = 0$ we obtain $F = 0$, $F_x = 0$, $F_y = 0$. Here our theorem fails, as might be expected from the fact that two different branches of the lemniscate pass through the origin. However, at all points of the curve where $y \neq 0$, our rule applies, and the derivative of the function $y = f(x)$ is given by

$$y' = -\frac{F_x}{F_y} = -\frac{4x(x^2 + y^2) - 4a^2x}{4y(x^2 + y^2) + 4a^2y}.$$

We can obtain important information about the curve from this equation, without using the explicit expression for y. For example, maxima or minima might occur where $y' = 0$, that is, for $x = 0$ or for $x^2 + y^2 = a^2$. From the equation of the lemniscate, $y = 0$ when $x = 0$; but at the origin there is no extreme value (cf. Fig. 1.S.3, Volume I, p. 103). The two equations therefore give the four points $\left(\pm \frac{a}{2}\sqrt{3}, \pm \frac{a}{2}\right)$ as the maxima and minima.

2. The *folium of Descartes* has the equation

$$F(x, y) = x^3 + y^3 - 3axy = 0$$

(cf. Fig 3.3), with awkward explicit solutions. At the origin, where the curve intersects itself, our rule again fails, since at that point $F = F_x = F_y = 0$. For all points at which $y^2 \neq ax$ we have

$$y' = -\frac{F_x}{F_y} = -\frac{x^2 - ay}{y^2 - ax}.$$

Accordingly, there is a zero of the derivative when $x^2 - ay = 0$ or, if we use the equation of the curve, when

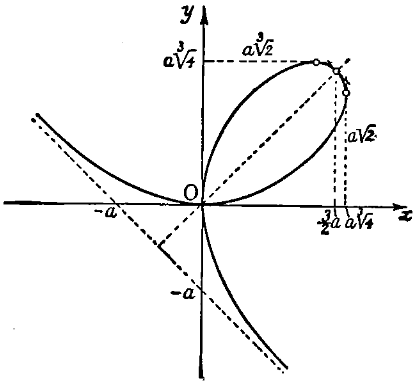

Figure 3.3 Folium of Descartes.

$$x = a\sqrt[3]{2}, \quad y = a\sqrt[3]{4}.$$

Exercises 3.1c

1. Prove that the following equations have unique solutions for y near the points indicated:
   (a) $x^2 + xy + y^2 = 7 \qquad (2, 1)$
   (b) $x \cos xy = 0 \qquad (1, \pi/2)$
   (c) $xy + \log xy = 1 \qquad (1, 1)$
   (d) $x^5 + y^5 + xy = 3 \qquad (1, 1).$
2. Find the first derivatives of the solutions in Exercise 1 and give their values at the indicated points.
3. Find the second derivatives of the solutions in Exercise 1 and give their values at the indicated points.
4. Which of the implicitly defined functions of Exercise 1 are convex at the indicated points.
5. Find the maximum and minimum values of the function y that satisfies the equation $x^2 + xy + y^2 = 27$.
6. Let $f_y(x, y)$ be continuous on a neighborhood of the point (x_0, y_0). Show that the equation

$$y = y_0 + \int_{x_0}^{x} f(\xi, y)d\xi$$

determines y as a function of x in some interval about $x = x_0$.

d. *Proof of the Implicit Function Theorem*

Existence of the implicit function follows directly from the intermediate value theorem (see Volume I, p. 44). Assume that $F(x, y)$ is defined and has continuous first derivatives in a neighborhood of the point (x_0, y_0), and let

$$F(x_0, y_0) = 0, \quad F_y(x_0, y_0) \neq 0.$$

Without loss of generality we assume that $m = F_y(x_0, y_0) > 0$. Otherwise, we merely replace the function F by $-F$, which leaves the points described by the equation $F(x, y) = 0$ unaltered. Since $F_y(x, y)$ is continuous, we can find a rectangle R with center (x_0, y_0) and so small that R lies completely in the domain of F and $F_y(x, y) > m/2$ throughout R. Let R be the rectangle

$$x_0 - a \leqq x \leqq x_0 + a, \quad y_0 - \beta \leqq y \leqq y_0 + \beta$$

(see Fig. 3.4). Since $F_x(x, y)$ also is continuous, we conclude that F_x

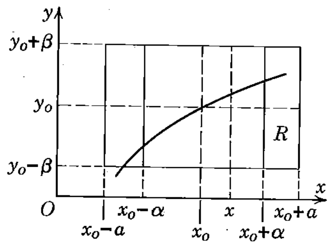

Figure 3.4

is bounded in R. Thus, there exist positive constants m, M such that

$$(6) \qquad F_y(x, y) > \frac{m}{2}, \qquad |F_x(x, y)| \leqq M \qquad \text{for} \qquad (x, y) \text{ in } R.$$

For any fixed x between $x_0 - a$ and $x_0 + a$ the expression $F(x, y)$ is a continuous and monotonically increasing function of y for $y_0 - \beta \leqq y \leqq y_0 + \beta$. If

$$(7) \qquad F(x, y_0 + \beta) > 0, \qquad F(x, y_0 - \beta) < 0,$$

we can be sure that there exists a single value y intermediate between $y_0 - \beta$ and $y_0 + \beta$ at which $F(x, y)$ vanishes. For the given x the equation $F(x, y)$ will then have a single solution $y = f(x)$ for which

$$y_0 - \beta < y < y_0 + \beta.$$

To prove (7), we observe that by the mean value theorem

$$F(x, y_0) = F(x, y_0) - F(x_0, y_0) = F_x(\xi, y_0)(x - x_0).,$$

where ξ is intermediate between x_0 and x. Hence, if α denotes a number between 0 and a, we have

$$|F(x, y_0)| \leqq |F_x(\xi, y_0)|\,|x - x_0| \leqq M\alpha \qquad \text{for} \qquad |x - x_0| \leqq \alpha.$$

Similarly, it follows from $F_y > m/2$ that

$$F(x, y_0 + \beta) = [F(x, y_0 + \beta) - F(x, y_0)] + F(x, y_0) > \frac{1}{2} m\beta - M\alpha,$$

$$F(x, y_0 - \beta) = -[F(x, y_0) - F(x, y_0 - \beta)] + F(x, y_0) < -\frac{1}{2} m\beta + M\alpha.$$

Thus, the inequalities (7) hold for any x in the interval $x_0 - \alpha \leqq x \leqq$

$x_0 + \alpha$ provided we take α so small that $\alpha \leqq a$ and $\alpha < m\beta/2M$.

For any x with $|x - x_0| \leqq \alpha$ this proves existence and uniqueness of a solution $y = f(x)$ of the equation $F(x, y) = 0$ such that $|y - y_0| \leqq \beta$ and $F_y(x, y) > m/2 > 0$. For $x = x_0$ the equation $F(x, y) = 0$ has the solution $y = y_0$ corresponding to our initial point. Since y_0 certainly lies between $y_0 - \beta$ and $y_0 - \beta$, we see that $f(x_0) = y_0$. Continuity and differentiability of $f(x)$ now follow from the mean value theorem for functions of several variables applied to $F(x, y)$ [see (33) p. 67]. Let x and $x + h$ be two values between $x_0 - \alpha$ and $x_0 + \alpha$. Let $y = f(x)$ and $y + k = f(x + h)$ be the corresponding values of f where y and $y + k$ lie between $y_0 - \beta$ and $y_0 + \beta$. Then $F(x, y) = 0$, $F(x + h, y + k) = 0$. It follows that

$$\begin{aligned} 0 &= F(x + h, y + k) - F(x, y) \\ &= F_x(x + \theta h, y + \theta k)\, h + F_y(x + \theta h, y + \theta k) k, \end{aligned}$$

where θ is a suitable intermediate value between 0 and 1.[1] Using $F_y \neq 0$, we can divide by F_y and find that

$$\frac{k}{h} = -\frac{F_x(x + \theta h, y + \theta k)}{F_y(x + \theta h, y + \theta k)}. \tag{8}$$

Since $|F_x| \leqq M$, $|F_y| > m/2$ for all points of our rectangle, we find that the right-hand side is bounded by $2M/m$. Thus

$$|k| \leqq \frac{2M}{m} |h|.$$

Hence, $k = f(x + h) - f(x) \to 0$ for $h \to 0$, which shows that $y = f(x)$ is a continuous function. We conclude from (8) that for fixed x and for $y = f(x)$,

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = -\lim_{h \to 0} \frac{F_x(x + \theta h, y + \theta k)}{F_x(x + \theta h, y + \theta k)} = -\frac{F_x(x, y)}{F_y(x, y)}.$$

This establishes the differentiability of f and at the same time yields formula (4) for the derivative.

The proof hinges on the assumption $F_y(x_0, y_0) \neq 0$, from which we could conclude that F_y is of constant sign in a sufficiently small

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[1] Observe that the mean value theorem can be applied here, since the segment joining any two points of the rectangle $|x - x_0| \leqq \alpha$, $|y - y_0| \leqq \beta$ lies wholly within the rectangle.

neighborhood of (x_0, y_0) and that $F(x, y)$ for fixed x is a monotone function of y.

The proof merely tells us that the function $y = f(x)$ exists. It is a typical example of a pure "existence theorem," in which the practical possibility of calculating the solution is not considered. Of course, we could apply any of the numerical methods discussed in Volume I (pp. 494 ff.) to approximate the solution y of the equation $F(x, y) = 0$ for fixed x.

Exercises 3.1d

1. Give an example of a function $f(x, y)$ such that (a) $f(x, y) = 0$ can be solved for y as a function of x near $x = x_0$, $y = y_0$, and (b) $f_y(x_0, y_0) = 0$.
2. Give an example of an equation $F(x, y) = 0$ that can be solved for y as a function $y = f(x)$ near a point (x_0, y_0), such that f is not differentiable at x_0.
3. Let $\phi(x)$ be defined for all real values of x. Show that the equation $F(x, y) = y^3 - y^2 + (1 + x^2)\, y - \phi(x) = 0$ defines a unique value of y for each value of x.

e. *The Implicit Function Theorem for More Than Two Independent Variables*

The implicit function theorem can be extended to a function of several independent variables as follows:

Let $F(x, y, \ldots, z, u)$ be a continuous function of the independent variables $x, y, \ldots z, u$, with continuous partial derivatives $F_x, F_y, \ldots, F_z, F_u$. Let $(x_0, y_0, \ldots, z_0, u_0)$ be an interior point of the domain of definition of F, for which

$$F(x_0, y_0, \ldots, z_0, u_0) = 0 \qquad \text{and} \qquad F_u(x_0, y_0, \ldots, z_0, u_0) \neq 0.$$

Then we can mark off an interval $u_0 - \beta \leqq u \leqq u_0 + \beta$ about u_0 and a rectangular region R containing $(x_0, y_0, \ldots, z_0)$ in its interior such that for every $(x, y, \ldots, z)$ in R, the equation $F(x, y, \ldots, z, u) = 0$ is satisfied by exactly one value of u in the interval $u_0 - \beta \leqq u \leqq u_0 + \beta$.[1] *For this value of u, which we denote by $u = f(x, y, \ldots, z)$, the equation*

$$F(x, y, \ldots, z, f(x, y, \ldots, z)) = 0$$

holds identically in R; in addition,

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[1] The value β and the rectangular region R are not determined uniquely. The assertion of the theorem is valid if β is any sufficiently small positive number and if we choose R (depending on β) sufficiently small.