Math S–101. Worksheet 15.
Topological Equivalence

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Injections, Bijections, and Surjections

- A function $f : X \to Y$ is *injective* or *one-to-one* if for any $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies that $x_1 = x_2$.
- A function $f : X \to Y$ is *surjective* or *onto* if for any $y \in Y$, there exists an $x \in X$ such that $f(X) = y$.
- A function $f : X \to Y$ that is both injective and surjective is *bijective*.

Let $f : X \to Y$ be a map and suppose there exists a map $g : Y \to X$ such that $f(g(y)) = y$ for all $y \in Y$ and $g(f(x)) = x$ for all $x \in X$. Then g is the inverse map and is unique. We will denote g by f^{-1}.

The inverse function $f^{-1} : Y \to X$ exists if and only if $f : X \to Y$ is a bijection.

Topological Equivalence

Let $(X, \mathbf{K}_X)$ and $(Y, \mathbf{K}_Y)$ be topological spaces. A function $h : X \to Y$ is a *homeomorphism* if it is bijective and if for any $A \subset Y$,

$$\mathbf{K}_Y(A) = h(\mathbf{K}_X(h^{-1}(A))).$$

If there exists a homeomorphism from $(X, \mathbf{K}_X)$ to $(Y, \mathbf{K}_Y)$, then we say that these two topological spaces are *homeomorphic* or *topologically equivalent* and write

$$(X, \mathbf{K}_X) \approx (Y, \mathbf{K}_Y).$$

Theorem 1 *If* $(X, \mathbf{K}_X) \approx (Y, \mathbf{K}_Y)$*, then* $(X, \mathbf{K}_X)$ *is connected if and only if* $(Y, \mathbf{K}_Y)$ *is connected.*

Theorem 2 (Homeomorphism Theorem) *If* $(X, \mathbf{K}_X) \approx (Y, \mathbf{K}_Y)$ *are topological spaces, then* $h : X \to Y$ *is a homeomorphism if and only if there exists a* $g : Y \to X$ *such that*

1. $g \circ f$ *is the identity on* X*.*
2. $f \circ g$ *is the identity on* Y*.*
3. f *and* g *are both continuous.*

Topological Invariants

A *topological invariant* is a property that does not change under homeomorphism. For example, if $(X, \mathbf{K}_X) \approx (Y, \mathbf{K}_Y)$, then X is connected if and only if Y is connected.

The Fixed Point Property

A space $(X, \mathbf{K})$ has the fixed point property if every continuous function $f : X \to X$ has a fixed point. The fixed point property is a topological invariant.

Some Problems

1. Show that the inverse map is unique if it is defined.
2. Prove the inverse function $f^{-1} : Y \to X$ exists if and only if $f : X \to Y$ is a bijection.
3. If $(X, \mathbf{K}_X) \approx (Y, \mathbf{K}_Y)$, show that $(X, \mathbf{K}_X)$ is connected if and only if $(Y, \mathbf{K}_Y)$ is connected.
4. Show that the fixed point property is a topological invariant.