Math S–101. Worksheet 14.
The Brouwer Fixed Point Theorem in One Dimension

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The Intermediate Value Theorem

Lemma 1 *If S is a connected subset of $\mathbb{R}$, then S is an interval.*

Theorem 2 (Intermediate Value Theorem) *Suppose that $a \leq b$ and the function $f : [a, b] \to \mathbb{R}$ is continuous. If y is between $f(a)$ and $f(b)$, then there exists a $c \in [a, b]$ such that $f(c) = y$.*

The Brouwer Fixed Point Theorem in One Dimension

Theorem 3 *Suppose that $a \leq b$ and $f : [a, b] \to [a, b]$ is continuous. Then f has a fixed point.*

Some Problems

1. Prove Lemma 1. *Hint*: If S is not an interval, show that S must be disconnected.

2. Supply the reasons for each statement in the following proof of the Intermediate Value Theorem.

   (a) The interval $[a, b]$ is connected.

   (b) The set $f([a, b])$ is connected.

   (c) The set $f([a, b])$ is an interval.

   (d) Since $f(a)$ and $f(b)$ are in $f([a, b])$, it follows that y is in $f([a, b])$.

   (e) There exists a $c \in [a, b]$ such that $f(c) = y$.

3. Supply the reasons for each statement in the following proof of the Brouwer Fixed Theorem for one dimension.

   (a) Given $f : [a, b] \to [a, b]$, define

   $$g(x) = f(x) - x.$$

   The function g is continuous from $[a, b]$ to $\mathbb{R}$.

   (b) $g(a) \geq 0$ and $g(b) \leq 0$.

   (c) There exists $x \in [a, b]$ such that $g(x) = 0$.

   (d) The point x is a fixed point for f.