Math S–101. Worksheet 12.
Continuous Functions

T. Judson

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Induced Set Functions

Suppose that $f : X \to Y$ and A, B are subsets of X and V, W subsets of Y. Then

1. $f(\emptyset) = \emptyset$
2. $f(A \cup B) = f(A) \cup f(B)$
3. $A \subset B \Longrightarrow f(A) \subset f(B)$
4. $f^{-1}(\emptyset) = \emptyset$
5. $f^{-1}(V \cup W) = f^{-1}(V) \cup f^{-1}(W)$
6. $V \subset W \Longrightarrow f^{-1}(V) \subset f^{-1}(W)$
7. $f^{-1}(V \cap W) = f^{-1}(V) \cap f^{-1}(W)$
8. $f^{-1}(V \setminus W) = f^{-1}(V) \setminus f^{-1}(W)$
9. $f^{-1}(Y) = X$
10. $A \subset f^{-1}(f(A))$
11. $W \supset f(f^{-1}(W))$

Continuous Functions

Let $(X, \mathbf{K}_X)$ and $(Y, \mathbf{K}_Y)$ be topological spaces. A function $f : X \to Y$ is *continuous* if it satisfies one of the following equivalent statements.

1. For every $A \subset X$, $f(\mathbf{K}_X(A)) \subset \mathbf{K}_Y(f(A))$.
2. If U is an open subset of Y, then $f^{-1}(U)$ is an open subset of X.
3. If E is an closed subset of Y, then $f^{-1}(E)$ is an closed subset of X.

The composition of continuous functions is continuous.

Continuous Functions and Connected Sets

Let $(X, \mathbf{K}_X)$ and $(Y, \mathbf{K}_Y)$ be topological spaces. If X is connected and the function $f : X \to Y$ is continuous with $f(X) = Y$, then Y is connected.

The Relative Topology

Let $(X, \mathbf{K}_X)$ be a topological space and $W \subset X$. The *relative closure operator*, $\mathbf{K}_W$ on W is defined by

$$\mathbf{K}_W(A) = W \cap \mathbf{K}_X(A)$$

for all $A \subset W$. We say that $\mathbf{K}_W$ is the closure operator on W *induced* by $\mathbf{K}_X$.

The Brouwer Fixed Point Theorem

Define the *closed unit ball* to be

$$B^n = \{x \in \mathbb{R}^n | d(x, 0) \leq 1\}.$$

If $f : B^n \to B^n$ is continuous, then f has a fixed point. That is, there exists an $x \in B^n$ such that $f(x) = x$.