Math S–101. Worksheet 11.
Completeness of the Real Numbers

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Summer 2006

Archimedean Fields

An ordered field $\mathbb{F}$ containing $\mathbb{Z}$ is *Archimedean* if for all $\epsilon \geq 0$ in $\mathbb{F}$, $\epsilon < 1/n$ for all $n \in \mathbb{N}$ implies $\epsilon = 0$. The rational numbers, $\mathbb{Q}$ is an example of an Archimedean field. However, the rationals do not satisfy the *Principle of Nested Closed Intervals*. Consider closed nested intervals that contain $\sqrt{2}$. Their intersections is $\sqrt{2}$, but this is not a rational number.

Least Upper Bound

Let $\mathbb{F}$ be an ordered field. We say that $b \in \mathbb{F}$ is an *upper bound* for a set $A \subset \mathbb{F}$ if $b \geq a$ for all $a \in A$. A *least upper bound* for a set A is a number M such that

- M is an upper bound for A.
- For any other upper bound d of A, $M \leq d$.

A field $\mathbb{F}$ satisfies the *Least Upper Bound Principle* if every subset A of $\mathbb{F}$ contains a least upper bound M in $\mathbb{F}$. We often write $M = \sup(A)$. We say that $\mathbb{F}$ is *complete*.

Dedekind Cuts

A subset A of $\mathbb{Q}$ is called a *Dedekind cut* if

- A is not empty and $A \neq \mathbb{Q}$.
- If $r \in A$ and $s \in \mathbb{Q}$ with $s < r$, then $s \in A$.
- A does not contain a largest element.

The set of all Dedekind cuts is the smallest field containing a copy of the rationals that is ordered, Archimedean, and complete. We call this field the real numbers, $\mathbb{R}$.