Math S–101. Worksheet 6.
Sets (II): Partitions and Equivalence Relations

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Equivalence Relations

An ***equivalence relation*** on a set X is a relation $R \subset X \times X$ such that

- $(x, x) \in R$ for all $x \in X$ (***reflexive property***);
- $(x, y) \in R$ implies $(y, x) \in R$ (***symmetric property***);
- (x, y) and $(y, z) \in R$ imply $(x, z) \in R$ (***transitive property***).

Given an equivalence relation R on a set X, we usually write $x \sim y$ instead of $(x, y) \in R$. If the equivalence relation already has an associated notation such as $=$, $\equiv$, or $\cong$, we will use that notation.

Partitions

A ***partition*** $\mathcal{P}$ of a set X is a collection of nonempty sets $X_1, X_2, \ldots$ such that $X_i \cap X_j = \emptyset$ for $i \neq j$ and $\bigcup_k X_k = X$. Let $\sim$ be an equivalence relation on a set X and let $x \in X$. Then $[x] = \{y \in X : y \sim x\}$ is called the ***equivalence class*** of x.

Theorem 1 *Given an equivalence relation $\sim$ on a set X, the equivalence classes of X form a partition of X. Conversely, if $\mathcal{P} = \{X_i\}$ is a partition of a set X, then there is an equivalence relation on X with equivalence classes X_i.*

Corollary 2 *Two equivalence classes of an equivalence relation are either disjoint or equal.*

Problems

1. Determine whether or not the following relations are equivalence relations on the given set. If the relation is an equivalence relation, describe the partition given by it. If the relation is not an equivalence relation, state why it fails to be one.

   (a) $x \sim y$ in $\mathbb{R}$ if $x \geq y$

   (b) $m \sim n$ in $\mathbb{Z}$ if $mn > 0$

   (c) $x \sim y$ in $\mathbb{R}$ if $|x - y| \leq 4$

   (d) $m \sim n$ in $\mathbb{Z}$ if $m \equiv n \pmod 6$

2. Find the error in the following argument by providing a counterexample. "The reflexive property is redundant in the axioms for an equivalence relation. If $x \sim y$, then $y \sim x$ by the symmetric property. Using the transitive property, we can deduce that $x \sim x$."

3. **Projective Real Line.** Define a relation on $\mathbb{R}^2 \setminus (0,0)$ by letting $(x_1, y_1) \sim (x_2, y_2)$ if there exists a nonzero real number λ such that $(x_1, y_1) = (\lambda x_2, \lambda y_2)$. Prove that $\sim$ defines an equivalence relation on $\mathbb{R}^2 \setminus (0,0)$. What are the corresponding equivalence classes? This equivalence relation defines the projective line, denoted by $\mathbb{P}(\mathbb{R})$, which is very important in geometry.