Math S–101. Worksheet 5.
Sets (I): Set Operations and Maps

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Sets

A set A is a ***subset*** of B, written $A \subset B$ or $B \supset A$, if every element of A is also an element of B. Trivially, every set is a subset of itself. A set B is a ***proper subset*** of a set A if $B \subset A$ but $B \neq A$. If A is not a subset of B, we write $A \not\subset B$. Two sets are ***equal***, written $A = B$, if we can show that $A \subset B$ and $B \subset A$. The set with no elements in it is called the ***empty set*** and is denoted by $\emptyset$. Note that the empty set is a subset of every set.

To construct new sets out of old sets, we can perform certain operations: the ***union*** $A \cup B$ of two sets A and B is defined as

$$A \cup B = \{x : \ x \in A \text{ or } x \in B\};$$

the ***intersection*** of A and B is defined by

$$A \cap B = \{x : \ x \in A \text{ and } x \in B\}.$$

When two sets have no elements in common, they are said to be ***disjoint***. Two sets A and B are disjoint exactly when $A \cap B = \emptyset$. Sometimes we will work within one fixed set U, called the ***universal set***. For any set $A \subset U$, we define the ***complement*** of A, denoted by A', to be the set

$$A' = \{x : \ x \in U \text{ and } x \notin A\}.$$

We define the ***difference*** of two sets A and B to be

$$A \setminus B = A \cap B' = \{x : \ x \in A \text{ and } x \notin B\}.$$

Maps

Given two sets A and B, we can define a new set $A \times B$, called the ***Cartesian product*** of A and B, as a set of ordered pairs. That is,

$$A \times B = \{(a, b) : \ a \in A \text{ and } b \in B\}.$$

Subsets of $A \times B$ are called ***relations***. We will define a ***map*** or ***function*** $f \subset A \times B$ from a set A to a set B to be the special type of relation in which for each element $a \in A$ there is a unique element $b \in B$ such that $(a, b) \in f$; another way of saying this is that for every element in A, f assigns a unique element in B. We usually write $f : A \to B$ or $A \xrightarrow{f} B$. Instead of writing down ordered pairs $(a, b) \in A \times B$, we write $f(a) = b$ or $f : a \mapsto b$. The set A is called the ***domain*** of f and

$$f(A) = \{f(a) : a \in A\} \subset B$$

is called the ***range*** or ***image*** of f.

A relation is ***well-defined*** if each element in the domain is assigned to a *unique* element in the range.

If $f : A \to B$ is a map and the image of f is B, i.e., $f(A) = B$, then f is said to be ***onto*** or ***surjective***. A map is ***one-to-one*** or ***injective*** if $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$. Equivalently, a function is one-to-one if $f(a_1) = f(a_2)$ implies $a_1 = a_2$. A map that is both one-to-one and onto is called ***bijective***.

Given two functions, we can construct a new function by using the range of the first function as the domain of the second function. Let $f : A \to B$ and $g : B \to C$ be mappings. Define a new map, the ***composition*** of f and g from A to C, by $(g \circ f)(x) = g(f(x))$.

Problems

1. Prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
2. Prove $A \subset B$ if and only if $A \cap B = A$.
3. Prove $(A \cap B)' = A' \cup B'$.
4. (a) Define a function $f : \mathbb{N} \to \mathbb{N}$ that is one-to-one but not onto.
   (b) Define a function $f : \mathbb{N} \to \mathbb{N}$ that is onto but not one-to-one.
5. Let $f : A \to B$ and $g : B \to C$ be maps.
   (a) If f and g are both one-to-one functions, show that $g \circ f$ is one-to-one.
   (b) If $g \circ f$ is onto, show that g is onto.
   (c) If $g \circ f$ is one-to-one, show that f is one-to-one.
   (d) If $g \circ f$ is one-to-one and f is onto, show that g is one-to-one.
   (e) If $g \circ f$ is onto and g is one-to-one, show that f is onto.