Math S–101. Worksheet 3.
The Integers (II)

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The Division Algorithm

- **The Division Algorithm.** Let a and b be integers, with $b > 0$. Then there exist unique integers q and r such that

  $$a = bq + r$$

  where $0 \leq r < b$.

- **Theorem.** Let a and b be nonzero integers. Then there exist integers r and s such that

  $$\gcd(a, b) = ar + bs.$$

  Furthermore, the greatest common divisor of a and b is unique. In particular, if a and b be two integers that are relatively prime, there exist integers r and s such that $ar + bs = 1$.

- **Euclid's Theorem on Prime Numbers.** There exist an infinite number of primes.

- **Fundamental Theorem of Arithmetic** Let n be an integer such that $n > 1$. Then

  $$n = p_1 p_2 \cdots p_k,$$

  where $p_1, \ldots, p_k$ are primes (not necessarily distinct). Furthermore, this factorization is unique; that is, if

  $$n = q_1 q_2 \cdots q_l,$$

  then $k = l$ and the q_i's are just the p_i's rearranged.

Problems

1. For each of the following pairs of numbers a and b, calculate $\gcd(a, b)$ and find integers r and s such that $\gcd(a, b) = ra + sb$.

   (a) 234 and 165

   (b) 471 and 562

2. Let a and b be nonzero integers. If there exist integers r and s such that $ar + bs = 1$, show that a and b are relatively prime.

3. Let $x, y \in \mathbb{N}$ be relatively prime. If xy is a perfect square, prove that x and y must both be perfect squares.

4. Using the division algorithm, show that every perfect square is of the form $4k$ or $4k + 1$ for some nonnegative integer k.

5. Prove that there are an infinite number of primes of the form $6n + 1$.