

# Math S–101. Worksheet 1.

## Axioms of Integer Arithmetic

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### Axioms of Arithmetic

1. *Closure of Addition.* If  $a, b \in \mathbb{Z}$ , then  $a + b \in \mathbb{Z}$ .
2. *Closure of Multiplication.* If  $a, b \in \mathbb{Z}$ , then  $a \cdot b \in \mathbb{Z}$ .
3. *Additive Identity.* There exists a number  $0 \in \mathbb{Z}$  such that  $0 + a = a$  for all  $a \in \mathbb{Z}$ .
4. *Multiplicative Identity.* There exists a number  $1 \in \mathbb{Z}$  such that  $1 \cdot a = a$  for all  $a \in \mathbb{Z}$ .
5. *Associativity of Addition.* For all  $a, b, c \in \mathbb{Z}$ ,

$$a + (b + c) = (a + b) + c.$$

6. *Associativity of Multiplication.* For all  $a, b, c \in \mathbb{Z}$ ,

$$a(bc) = (ab)c.$$

7. *Commutativity of Addition.* For all  $a, b \in \mathbb{Z}$ ,

$$a + b = b + a.$$

8. *Commutativity of Multiplication.* For all  $a, b \in \mathbb{Z}$ ,

$$ab = ba.$$

9. *Additive Inverses.* For every  $a \in \mathbb{Z}$ , there exists a  $-a \in \mathbb{Z}$  such that  $-a + a = 0$ .<sup>1</sup>

10. *Distribution.* For all  $a, b, c \in \mathbb{Z}$ ,

$$a(b + c) = ab + ac.$$

## Problems

We can derive the following from our axioms.

1. Show that  $(-1) \cdot (-1) = 1$
2. Show that  $(-a) \cdot b = -(ab)$  for all  $a, b$ .
3. Show that  $(-a) \cdot (-b) = ab$  for all  $a, b$ .
4. Show that  $0 \cdot a = 0$  for all  $a$ .
5. Show that zero is unique. That is, show that for any element  $0'$  such that  $0' + a = a$  for all  $a$ , it must be the case that  $0' = 0$ .
6. **Zero Divisors.** Prove that the following two statements are equivalent.
  - (a) If  $ab = 0$ , then either  $a = 0$  or  $b = 0$ .
  - (b) If  $ab = ac$  and  $a \neq 0$ , then  $b = c$ .

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<sup>1</sup>If we change to the rational numbers  $\mathbb{Q}$ , then we can state an axiom for a *multiplicative inverse*. That is, if  $a \in \mathbb{Q}$  is nonzero, then there exists an  $a^{-1} = 1/a \in \mathbb{Q}$  such that  $a^{-1}a = 1$ .