Assignment 6 Solution

August 14, 2006

Section 8.1, Exercise 6

Proving part (a): Let x be an element of A. Then $f(x) \in f(A)$ by definition of a function. This implies $x \in f^{-1}(f(A))$ (since the image of x is in $f(A)$). So $A \subset f^{-1}(f(A))$.
This is not true in the following example: X is the set $\{1, 2, 3\}$ and A is just $\{1\}$. Let f be a function mapping everything in X to the point $\{0\}$ in Y. Then the image of A is $\{0\}$ but the preimage of $f(A)$ is the set $\{1, 2, 3\}$.

Proving part (b): Let y be an element of $f(f^{-1}(A))$. Then there exists some x in $f^{-1}(A)$ such that $f(x) = y$. But if x is in $f^{-1}(A)$, then $f(x)$ must be in A. This means $y \in A$ and so $f(f^{-1}(A)) \subset A$.
This is not true in the following example: X is the set $\{1, 2, 3\}$ and Y is the set $\{4, 5, 6\}$. Let f be a function mapping everything in X to the point $\{4\}$ in Y and let $A = \{4, 5\}$. Then the preimage of A is $\{1, 2, 3\}$, but the image of $f^{-1}(A)$ is $\{4, 5, 6\}$.

Section 8.4, Exercise 1

You just need to prove that the function f that they've given is continuous but has no fixed point.

Continuity: consider a closed set C in X. There are four possible cases:

- If a and b are in C, then $f^{-1}C = A \cup B = X$ which is closed.
- If a is in C but b is not, then $f^{-1}C = B$ which is closed.
- If b is in C but a is not, then $f^{-1}C = A$ which is closed.
- If both a and b are not in C, then $f^{-1}C = \emptyset$ which is closed.

So the preimage of any closed set is closed. Hence the function is continuous.

Fixed point: Everything gets mapped to a or b, so those are the only possible fixed points. a is in A, so a is not in B because $A \cap B = \emptyset$. Thus the image of a is not a. We similarly check that the image of b is not b, and we conclude that the function has no fixed points.

This proves that a space that is not connected does not satisfy the fixed point property; so a space that does satisfy the fixed point property must be connected.

Section 8.4, Exercise 2

"$\Rightarrow$" Suppose X is disconnected, with some separation (A, B). Define a function $f(x)$ that maps A onto 0 and B onto 1. This is a surjective map since A and B are non-empty.
The only closed sets in $\{0, 1\}$ are $\emptyset$, $\{0\}$, $\{1\}$, and $\{0.1\}$. Their preimages are, respectively, $\emptyset$, A, B, and X. These sets are all closed. So the preimages of closed sets are closed and f is continuous.

"$\Leftarrow$" Suppose there is a continuous surjective function from X to $\{0, 1\}$. $\{0, 1\}$ is disconnected, so by the contrapositive of the connected image theorem, the statement "X is connected, $f : X \to \{0, 1\}$ is continuous, and $f(X) = \{0, 1\}$" is false. But we've already defined f as continuous with $f(X) = \{0, 1\}$. So the part of the statement that says "X is connected" must be false so that the overall statement can be false. Hence X is disconnected.

Section 9.2, Exercise 1

(a) Drawing inspiration from the next problem, we set $\delta = \epsilon = 1/17$. Then $d(x, y) < \delta \Rightarrow d(x, y) < \epsilon \Rightarrow d(f(x), f(y)) < \epsilon$ for any x, so the function is continuous. (You didn't even need to use $x = 3$ for this!)

(b) This one is more complicated. The main thing to remember when you do this is that if you start with $|f(x) - f(y)| < \epsilon$, instead of writing equations that are implications of this, you need to write equations that IMPLY this. (Because the definition of continuity starts with $d(x, y) < \delta$ and leads to a conclusion $d(f(x), f(y)) < \epsilon$.) In a lot of cases it won't matter because everything will work with iff, but not always.

$|f(x) - f(y)| < \epsilon$
By definition of x and ϵ, this is equivalent to $|2^2 + 2 + 1 - y^2 - y - 1| < 6$.
This is equivalent to $|6 - y^2 - y| < 6$.
$\Leftrightarrow -6 < 6 - y^2 - y < 6$
$\Leftrightarrow -12 < -y^2 - y < 0$
$\Leftrightarrow 0 < y^2 + y < 12$
This is true if $0 < y < 3$.
This is true if $-x < y - x < 3 - x$.
This is true if $-2 < y - x < 1$.
This is true if $-1 < y - x < 1$.
This last line is equivalent to $|y - x| < 1$, which is the same thing as saying $|x - y| < 1$.
So if we set $\delta = 1$, then this all works out.

Section 9.2, Exercise 2

Let $\delta = \epsilon$. Then $d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \delta \Rightarrow d(f(x), f(y)) < \epsilon$ for any x, y, so f is continuous.

Section 11.2, Exercise 3

(a) $f(x) = c + \frac{x-a}{b-a}(d - c)$. If x is in (a, b), then $0 < x - a < b - a$ so $0 < \frac{x-a}{b-a} < 1$. Then $0 < \frac{x-a}{b-a}(d - c) < d - c$ and $c < c + \frac{x-a}{b-a}(d - c) < d$. So $f(x) \in (c, d)$ which means that f maps (a, b) to (c, d).
$g(y) = a + \frac{y-c}{d-c}(b - a)$. If y is in (c, d), then $0 < y - c < d - c$ so $0 < \frac{y-c}{d-c} < 1$. Then $0 < \frac{y-c}{d-c}(b - a) < b - a$ and $a < a + \frac{y-c}{d-c}(b - a) < b$. So $g(y) \in (a, b)$ which means that g maps (c, d) to (a, b).
For $f(g(y))$ and $g(f(x))$ we just need two long equations:

$$f(g(y)) = c + \frac{a + \frac{y-c}{d-c}(b-a) - a}{b - a}(d - c) = y$$

$$g(f(x)) = a + \frac{c + \frac{x-a}{b-a}(d-c) - c}{d - c}(b - a) = x$$

(b) $f(x) = \frac{x}{1-|x|}$, $g(y) = \frac{y}{1+|y|}$
We need to show that g sends the real numbers into $(-1, 1)$:
Suppose there was some y such that $\frac{y}{1+|y|} \leq -1$. This would imply $y \leq -|y| - 1$ which is not possible. So $g(y) > -1$.

Suppose there was some y such that $\frac{y}{1+|y|} \geq 1$. This would imply $y \geq |y|+1$ which is not possible. So $g(y) < 1$.
This proves that the image of any real number is in the interval $(-1,1)$ which is what we needed to show.
Now we check $f(g(y))$ and $g(f(x))$:

$$g(f(x)) = \frac{\frac{x}{1-|x|}}{1+|\frac{x}{1-|x|}|}$$

$$= \frac{\frac{x}{1-|x|}}{1+\frac{|x|}{|1-|x||}}$$

$$= \frac{\frac{x}{1-|x|}}{1+\frac{|x|}{1-|x|}}$$

(This last step is because if x is in $(-1,1)$ then $1-|x|>0$.)

$$= \frac{\frac{x}{1-|x|}}{\frac{1-|x|+|x|}{1-|x|}}$$

$$= x$$

$$f(g(y)) = \frac{y1+|y|}{1-|\frac{y}{1+|y|}|}$$

$$= \frac{y1+|y|}{1-\frac{|y|}{|1+|y||}}$$

$$= \frac{y1+|y|}{1-\frac{|y|}{1+|y|}}$$

(This last step is because $1+|y|>0$.)

$$= \frac{y1+|y|}{\frac{1+|y|-|y|}{1+|y|}}$$

$$= y$$

Section 11.2, Exercise 6

We need to use the Homeomorphism Theorem by showing the following:
(a) h maps everything into the unit ball
(b) There is some function g that maps the unit ball back to $\mathbf{R}^n$
(c) This function g satisfies $f(g(y)) = y$ and $g(f(x)) = x$.

(a)$h(x) = \frac{x}{1+||x||}$
$\Rightarrow ||h(x)|| = ||\frac{x}{1+||x||}|| = \frac{||x||}{||(1+||x||)||} = \frac{||x||}{1+||x||} < 1$

So h maps everything into the unit ball.

(b) Consider the function $g(y) = \frac{y}{1-||y||}$. Since we never have $||y|| = 1$ in the unit ball, this map does indeed send everything from the unit ball into $\mathbf{R}^n$, and it's fairly obvious that this is well-defined.

(c) Again we just need to write out two long equations:

$$f(g(y)) = \frac{\frac{y}{1-||y||}}{1 + ||\frac{y}{1-||y||}||}$$

$$= \frac{\frac{y}{1-||y||}}{1 + \frac{||y||}{||(1-||y||)||}}$$

$$= \frac{\frac{y}{1-||y||}}{1 + \frac{||y||}{1-||y||}}$$

(This last step is because we have $||y|| < 1$, so $1 - ||y|| > 0$.)

$$= \frac{\frac{y}{1-||y||}}{\frac{1-||y||+||y||}{1-||y||}}$$

$$= y$$

$$g(f(x)) = \frac{\frac{x}{1+||x||}}{1 - ||\frac{x}{1+||x||}||}$$

$$= \frac{\frac{x}{1+||x||}}{1 - \frac{||x||}{||(1+||x||)||}}$$

$$= \frac{\frac{x}{1+||x||}}{1 - \frac{||x||}{1+||x||}}$$

(This last step is because $1 + ||x|| > 0$ for all x.)

$$= \frac{\frac{x}{1+||x||}}{\frac{1+||x||-||x||}{1+||x||}}$$

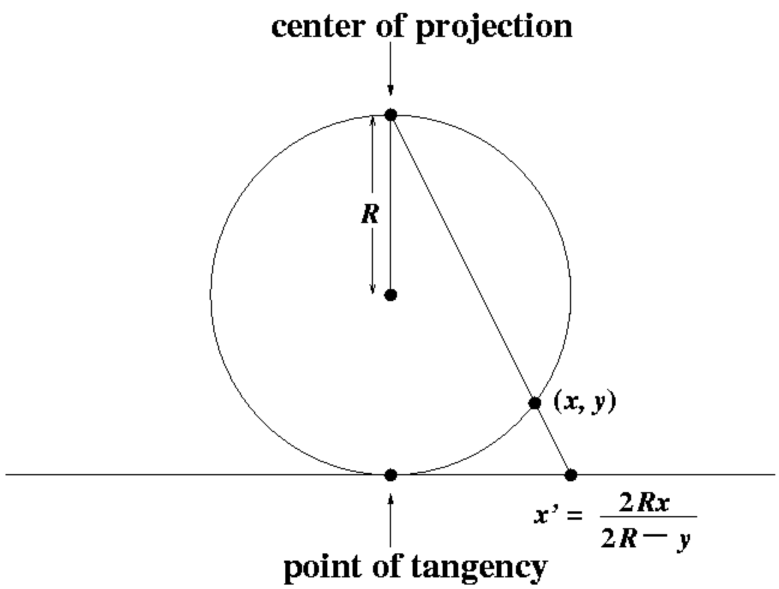
$$= x$$

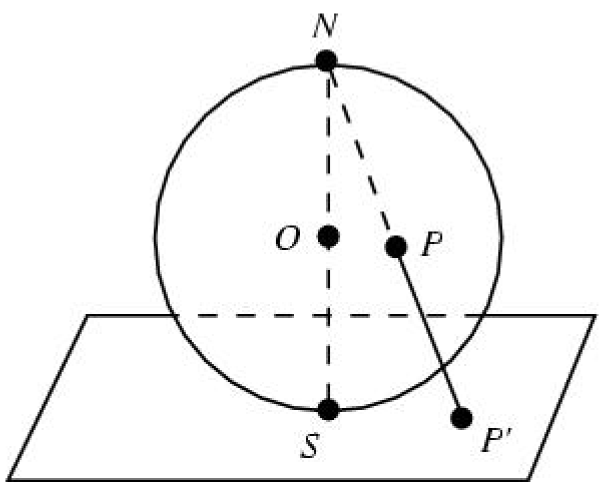
We've shown everything we needed to, so h is a homeomorphism.

Section 11.2, Exercise 7

The easiest way to do this is as follows. First off, it's pretty clear that no matter where I remove p, I can always rotate the sphere so that p is at the north pole. (Intuitively, this seems like it should be a homeomorphism, but this problem was not really big on proving stuff anyway as long as you explained precisely what you were doing.) Let's arrange the sphere in a coordinate system such that the south pole is at the origin and the north pole is at (0,0,2). Now for any point x on the sphere, I can take the line through p and x and I must get a unique point on the plane. It's clear that I will get the entire plane and that this is injective because for any point y of $\mathbf{R}^2$, I can draw the line through p and y and get a unique intersection on the sphere. So this defines a homeomorphism from the punctured unit sphere to the plane.

The following diagrams illustrate this construction (called stereographic projection). The first one (from wikipedia) is an example of what this would look like in 2-dimensions (i.e. showing the unit circle is homeomorphic to the real number line), while the second one (from mathworld.wolfram.com) shows how this works in 3-dimensions.





Section 11.3, Exercise 6

This problem basically boils down to 2 observations:
–If I have a letter with no holes in it (like C), and I draw a circle around it, then I can compress this circle onto C, and then compress C down to a point, and all of this is continuous. If I try to do this with a letter that like a hole in it (like O), all I can end up with is a circle shape. If I try to do this with a letter that has 2 holes (B), then I can only end up with two adjacent

circles. So it's pretty clear that the no-holed letters should be in a different equivalence class from the 1-holed letters, and they should all be separate from the 2-holed letter B.

–If I have a letter with one hole, I can continuously deform it into any other letter with one hole; similarly if I have a letter with no holes, then I can continuously deform it into any other letter with no holes. So it seems reasonable to put all the no-holed letters in the same equivalence class, and all the 1-holed letters in another.

This gives us the following partition of the alphabet:

–C,E,F,G,H,I,J,K,L,M,N,S,T,U,V,W,X,Y,Z

–A,D,O,P,Q,R

–B

Here are a few examples: