

Assignment 5 Solution

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Section 5.1, Exercise 1

(a) Let x and y be two rational numbers. We can write $x = a/b$ and $y = c/d$ for some a, b, c, d integers.

Then $x + y = a/b + c/d = ad/bd + bc/bd = (ad + bc)/bd$ where $(ad + bc)$ and bd are integers (closure of integers over addition and multiplication) and $bd \neq 0$ because $b \neq 0$ and $d \neq 0$. So $x + y$ is rational.

(b) Let x be a rational number and i be an irrational number. We can write $x = a/b$ for some integers a and b . Suppose the sum of x and i is a rational number – then we can write $a/b + i = c/d$ for some integers c and d . This implies $i = c/d - a/b = c/d + (-a)/b$. c/d and $(-a)/b$ are both rational, so their sum i must be rational from (a). This is a contradiction because we started out with i being irrational. So the sum of a rational and an irrational is an irrational number.

(c) There are a number of easy counter-examples. However many people lost points because they didn't bother proving that the numbers they were using were actually irrational. Using $\sqrt{2}$ saves you from proving it for one of your numbers:

We know that $\sqrt{2}$ is irrational from a prior problem set. Now we can easily show that $-\sqrt{2}$ is irrational. We know that $\sqrt{2} + (-\sqrt{2}) = 0$. 0 is clearly rational. If $-\sqrt{2}$ is rational, then we've just written *irrational* + *rational* = *rational* which is not true from (b). So $-\sqrt{2}$ must be irrational. So we've just found two irrational numbers whose sum is rational.

Section 5.1, Exercise 2

Proof by contradiction: suppose $\sqrt{6}$ is rational. Then we can write $\sqrt{6} = a/b$ for some integers a and b such that $\gcd(a, b) = 1$. So we get $6b^2 = a^2$. Since

6 is even, this means a^2 is even and hence a is even.

$\Rightarrow a = 2k$ for some integer k

$\Rightarrow a^2 = 4k^2$

$\Rightarrow 6b^2 = 4k^2$

$\Rightarrow 3b^2 = 2k^2$

Since 3 is clearly not even, this implies b^2 is even and hence b is even. But then $\gcd(a, b) = 2 \neq 1$ which contradicts the initial assumption. So $\sqrt{6}$ is irrational.

Section 6.2, Exercise 6

A lot of people used contrapositive or contradiction for this problem, but it's nice to use a direct proof when you can, as was the case here:

Let $x = \sup(A)$. Then for any $\epsilon > 0$, $x - \epsilon$ is not an upper bound of A (by definition of least upper bound). So there must exist some point a in A that is located between $x - \epsilon$ and x . In other words, $x - \epsilon < a \leq x$.

$\Rightarrow x - \epsilon < a < x + \epsilon$.

This is true for any $\epsilon > 0$, so this means any open ball around x intersects A , which implies $x \in \mathbf{KA}$.

Section 6.2, Exercise 8

To show that the rationals are dense in the reals, it suffices to show that any real number is in the closure of the rationals. (We don't need to show that the closure of the rationals is in the reals, because that's automatic if we take the reals to be our entire space.)

Let x be a real number. x is either rational or irrational. If x is rational it's immediate that x is in \mathbf{KQ} . If x is irrational, then we can write $x = x_1.x_2x_3x_4\dots x_i x_{i+1}\dots$ where the x_i 's are non-repeating. Now consider an open ball of radius $\epsilon > 0$ about x . We know that ϵ is greater than $1/n$ for some n (Archimedean property). So this means that there is some k such that $\epsilon > 1/10^k$. So the rational number $y = x_1.x_2x_3x_4\dots x_k x_{k+1}$ is in the ball about x . This holds for any x and any ϵ . So any ball about an irrational always intersects \mathbf{Q} . Therefore any irrational number is in the closure of the rationals, so the rationals are dense in the reals.

A similar proof will show that the irrationals are dense. All we need to show is that any rational is in the closure of the irrationals:

Let x be a rational number. We can write $x = x_1.x_2x_3x_4\dots x_n$ or $x = x_1.x_2x_3x_4\dots(\textit{infinitely repeating decimals})\dots$. Consider an open ball of ra-

dus $\epsilon > 0$ about x . There is some k such that $\epsilon > 1/10^k$. So the following irrational number is in the open ball: $y = x_1.x_2x_3x_4\dots x_kx_{k+1}01001000100001\dots$ (It's clear that $01001000100001\dots$ is a non-repeating sequence.) This holds for any x and any ϵ . So any ball about a rational always intersects the set of irrationals. Therefore any rational number is in the closure of the irrationals, so the irrationals are dense in the reals.

The reason we cannot define $\mathbf{R} = \mathbf{KQ}$ is because when we take the closure of a set, we always take it as a subset of some larger space. So we would need to already know that the reals exist in order to define this, and we would already need to know what the reals are.

\mathbf{R} is an example of a separable space, because it has a countable subset \mathbf{Q} whose closure is all of \mathbf{R} .

The reals under the discrete metric are an example of a non-separable space. The discrete metric is the one we defined on the last problem set where $d(a, b) = 1$ if $a \neq b$. You may remember that this metric induced a closure operator $\mathbf{K}A = A$. So any countable subset S of the reals under this metric will have $\mathbf{K}S = S \neq \mathbf{R}$. So this is a non-separable space.