# Assignment 4 Solution

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## Section 2.4, Exercise 2

(b) C1: Let $x$ be in $A$. Then $x$ is not in any $V$ such that $V \cap A = \emptyset$. So $x$ is not in $\cup V$. So $x$ is in $X - \cup V = \mathbf{K}A$. Hence $A \subset \mathbf{K}A$.

C2:

$$\mathbf{K}A \cup \mathbf{K}B = (X - \bigcup_{V \in T | V \cap A = \emptyset} V) \cup (X - \bigcup_{W \in T | W \cap B = \emptyset} W)$$

$$= (\bigcup_{V \in T | V \cap A = \emptyset} V)^c \cup (\bigcup_{W \in T | W \cap B = \emptyset} W)^c$$

$$= [(\bigcup_{V \in T | V \cap A = \emptyset} V) \cap (\bigcup_{W \in T | W \cap B = \emptyset} W)]^c$$

$$= (\bigcup_{Y \in T | Y \cap A = \emptyset and Y \cap B = \emptyset} Y)^c$$

[This last step basically says if I put together every open set that doesn't intersect A, and I put together every open set that doesn't intersect B, and then I take the intersection of the two resulting open sets, then I get an open set that is the union of all the open sets that don't intersect $A$ or $B$.]

$$= (\bigcup_{Y \in T | Y \cap (A \cup B) = \emptyset} Y)^c$$

$$= X - \bigcup_{Y \in T | Y \cap (A \cup B) = \emptyset} Y$$

$$= \mathbf{K}(A \cup B)$$

C3: It's fairly easy to show equality here. We ask ourselves the following question: it's obvious that an open set that does not intersect $\mathbf{K}A$ cannot intersect $A$. Is it possible that there is an open set $O$ that does not intersect $A$ but that does intersect $\mathbf{K}A$? The answer is "No": such a set $O$ would intersect the boundary of $A$, which would mean that $O$ has a (partial) boundary. But this is inconsistent with $O$ being open. This all means that the collection of open sets that do not intersect $\mathbf{K}A$ is equal to the collection of open sets that do not intersect $A$. Now we can write the following equations:

$$\bigcup_{V\in T|V\cap\mathbf{K}A=\emptyset} V = \bigcup_{V\in T|V\cap A=\emptyset} V$$

$$X - \bigcup_{V\in T|V\cap\mathbf{K}A=\emptyset} V = X - \bigcup_{V\in T|V\cap A=\emptyset} V$$

$$\mathbf{K}\mathbf{K}A = \mathbf{K}A$$

C4: $V \cap \emptyset = \emptyset$ is true for all sets $V$. Therefore:

$$\mathbf{K}\emptyset = X - \bigcup_{V\in T|V\cap\emptyset=\emptyset} V$$

$$\mathbf{K}\emptyset = X - X = \emptyset$$

(c) Let's call $\mathbf{K}A$ the closure operator we started out with and $\mathbf{K}'A$ the closure operator defined in (b). We basically need to prove that $\mathbf{K}A = \mathbf{K}'A$ for all $A$.
Let $x$ be in $\mathbf{K}A$ for some $A$.
$\Leftrightarrow$ There exists $U \in T$ such that $x \in X - U$.
$\Leftrightarrow$ $x$ is not in $U$ such that $U \cap \mathbf{K}A = \emptyset$.
$\Leftrightarrow$ $x$ is not in $U$ such that $U \cap A = \emptyset$. (This is from what we did for C3 in (b).)
Now this is true for all such $U$, so this is equivalent to $x$ not in $\cup_{U\cap A=\emptyset} U$
$\Leftrightarrow x \in X - \cup U$
$\Leftrightarrow x \in \mathbf{K}'A$

(d) The procedure here is similar; we need to prove that an element of $T$ is also an element of $T'$ and vice-versa. (Let's say $T$ is the topology we

started with and $T'$ is the topology defined from part (a).)
Let $A$ be a set in $T$. The topology $T'$ says that a set is open in $T'$ if its complement is closed under the closure operator $\mathbf{K}$ derived from $T$. So let's check $\mathbf{K}(X - A)$:

$$\mathbf{K}(X - A) = X - \bigcup_{V \in T | V \cap (X-A) = \emptyset} V$$

Notice that $V \cap (X - A) = \emptyset$ is equivalent to saying $V$ is a subset of $A$. So when we take the union of all such $V$'s, we get back an open set (from iii) that must be a subset of $A$ – in other words, we get back $A$. This gives us $\mathbf{K}(X - A) = X - A$ and thus $A \in T'$.
Now let $A$ be a set in $T'$. Then we have $\mathbf{K}(X - A) = X - A$ by definition. We also have

$$\mathbf{K}(X - A) = X - \bigcup_{V \in T | V \cap (X-A) = \emptyset} V$$

This means we must have $A = \bigcup_{V \in T | V \cap (X-A) = \emptyset} V$. So $A$ is equal to the union of a bunch of open sets (sets in T). So $A$ is open (by iii). Hence $A \in T$.
$T \subset T'$ and $T' \subset T$ so $T = T'$.

### Section 2.5, Exercise 2

Suppose $1/4$ is not in $C$. This means $1/4$ is not in $A_n$ for some $n$.
$\Rightarrow 1/4$ is not in $1/3A_{n-1} \cup (2/3 + 1/3A_{n-1})$
$\Rightarrow 1/4$ is not in $1/3A_{n-1}$
$\Rightarrow 3/4$ is not in $A_{n-1}$
On the other hand, if $3/4$ is not in $A_n$, then $3/4$ is not in $1/3A_{n-1} \cup (2/3 + 1/3A_{n-1})$
$\Rightarrow 3/4$ is not in $(2/3 + 1/3A_{n-1})$
$\Rightarrow 9/4$ is not in $(2 + A_{n-1})$
$\Rightarrow 1/4$ is not in $A_{n-1}$

So if $1/4$ is not in $A_n$, then $3/4$ is not in $A_{n-1}$, which means $1/4$ is not in $A_{n-2}$, so $3/4$ is not in $A_{n-3}$ and so on...
But $n$ must be a nonnegative integer, so eventually we reach $A_{n-n} = A_0$. So eventually we get $1/4$ is not in $A_0$ or $3/4$ is not in $A_0$ (depending on whether the original $n$ was even or odd). Remember that $A_0 = [0, 1]$. $1/4$ and $3/4$ are certainly elements of $[0, 1]$, so they are elements of $A_0$. This contradicts the conclusion we just reached. We conclude that $1/4$ is in fact in $C$.

### Section 2.5, Exercise 4 AKA The Problem Of Doom

Let $x$ be an element of the Cantor Set. Then $x$ must be in every $A_n$. It's clear that we can determine "which side" of $A_n$ $x$ is on for every $n$. For example, 1/3 is on the "left-hand side" of $A_1$, because it's to the left of the split between 1/3 and 2/3. Then in $A_2$, 1/3 is on the "right-hand side", because the interval $[0, 1/3]$ splits into $[0, 1/9] \cup [2/9, 1/3]$. Then when we split the interval $[2/9, 1/3]$, 1/3 is again going to find itself on the right-hand side. It's clear that we can do this uniquely for any element of the Cantor Set – we can always say whether it's on the right-hand side or the left-hand side of each middle-third split, and it can never be on both at once. It's also clear that any two points $x$ and $y$ in $C$ must eventually find themselves on different sides of some split – if they did not, then that would mean that the interval $[x, y]$ was contained in $C$, which is impossible because $C$ has length 0.

Now let's map each point in $C$ to the following number:

$$\sum_{n=1}^{\infty} \frac{i_n}{10^n}$$

where $i_n$ is defined as follows:

- $i_n = 0$ if $x$ is on the left-hand side of $A_n$
- $i_n = 1$ if $x$ is on the right-hand side of $A_n$

Under this map, the point 1/3 gets sent to 0.01111111... because 1/3 is on the left-hand side of $A_1$ and is on the right-hand side of every subsequent $A_n$. The point 0 would get mapped to $0.0000... = 0$ and the point 1 would get mapped to 0.1111111....
You will notice that the digit before the decimal point is always 0. This is because of how we defined the map to basically never pay attention to where a point is in $A_0$ (since everything is in the same interval in $A_0$ anyway).

Now based on what we said at the end of the first paragraph, it is clear that the map we've defined is a bijection from $C$ to the set of all numbers in $[0, 1)$ whose only digits are 0 and 1. Again, any point in $C$ can be uniquely associated with such a number, and any such number (i.e. any sequence of "left-hand sides" and "right-hand sides") is associated with a unique element of $C$.

Now all we need to do is show that this strange set of "every number between 0 and 1 whose digits are 0 or 1" is uncountable. If you are a computer scientist, this should be very familiar!!
In fact, the set we've mapped onto is just the interval $[0, 1)$ written in binary (base 2). 0 in binary is equal to 0 in base 10, and 0.1111111... is just an infinite series $1/2 + 1/4 + 1/8 + ...$ which has limit 1 in base 10.
So this map hits every real number in $[0, 1)$. So we've found a bijection between the Cantor Set and the interval $[0, 1)$. This interval is uncountable. Therefore the Cantor Set is uncountable.
NOTE: This is just one of many fun ways of proving this. I encourage you to look them up online to see some different cool proofs. (You can also find a very easy proof of this if you first do problem 3 from the same chapter.)

## Section 3.3, Exercise 2

(a) We have $d(x_1, x_2) \leq r_2 - r_1$. Let $y$ be a point in $B(x_1; r_1)$. Then $d(x_1, ry) < r_1$. Using the triangle inequality:

$$d(x_2, y) \leq d(x_1, y) + d(x_1, x_2)$$

$$d(x_2, y) < r_1 + r_2 - r_1$$

$$d(x_2, y) < r_2$$

So $y$ is in $B(x_2; r_2)$ and we conclude $B(x_1; r_1) \subset B(x_2; r_2)$.

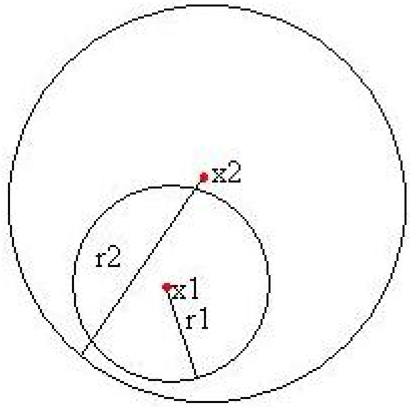
(b) We have $d(x_1, x_2) \geq r_1 + r_2$. Suppose $B(x_1; r_1) \cap B(x_2; r_2) \neq \emptyset$ and let $y$ be in the intersection. Then $d(x_1, y) < r_1$ and $d(x_2, y) < r_2$. By the triangle inequality:

$$d(x_1, x_2) \leq d(x_1, y) + d(x_2, y)$$

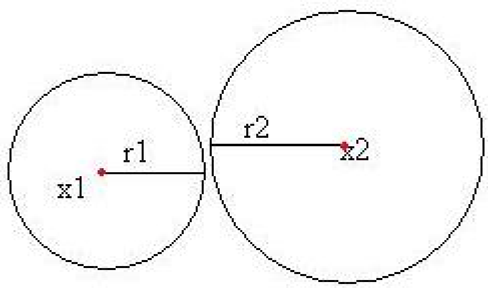
$$d(x_1, x_2) < r_1 + r_2$$

This contradicts our original statement.
Hence $B(x_1; r_1) \cap B(x_2; r_2) = \emptyset$

(c) This problem was somewhat ambiguous to what extent they wanted you to prove the converses. (Both converses are true.) Here's an example of a decent answer:
For (a), your diagram should look something like this:




For the proof: suppose $B(x_1; r_1) \subset B(x_2; r_2)$. Then the point $x_1$ must be at least at a distance $r_1$ from the boundary of $B(x_2; r_2)$, because if not there would be a point of $B(x_1; r_1)$ that would poke out of $B(x_2; r_2)$. So $d(x_1, y) \leq r_2 - r_1$.
For (b), your diagram might look like this:




Suppose $B(x_1; r_1) \cap B(x_2; r_2) = \emptyset$. The limiting case of this is when the boundaries of the balls touch at a point. Let's call that point $y$. If the boundaries touch at $y$, then $y$ must be on a line through $x_1$ and $x_2$ and must be between $x_1$ and $x_2$. (A notion of betweenness always exists on the real numbers to any dimension.) So now $d(x_1, x_2) = d(x_1, y) + d(x_2, y) = r_1 + r_2$. In all other cases, the distance will be greater because we will be able to find a point $y$ on the line such that $y$ is not contained in either ball and so is at a distance greater than $r_1$ or $r_2$ from $x_1$ and $x_2$ respectively. So we have $d(x_1, x_2) \geq r_1 + r_2$.
(This is not the most rigorous proof in the world, but part of the reason that the question was posed in such a strange way is that it would be hard to give a perfectly rigorous proof of these converses.)

### Section 3.3, Exercise 3

(NOTE: it is implicit in the wording of the problem that $d(x, y) = 0$ if $x = y$.)
We need to check the properties of a metric:
M1: $d(x, y) = 1 \geq 0$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$, so M1 is satisfied.
M2: If $x = y$, then $d(x, y) = 0 = d(y, x)$. If $x \neq y$, then $d(x, y) = 1 = d(y, x)$. So M2 is satisfied.
M3: We need to check $d(x, z) \leq d(x, y) + d(y, z)$ for all $x$, $y$, $z$. There are two cases here:

- If $x = z$, then $d(x, z) = 0$. Since all distances are nonnegative, we automatically have $0 \leq d(x, y) + d(y, z)$, and the triangle inequality holds.
- If $x \neq z$, then $d(x, z) = 1$. Notice now that we cannot have both $d(x, y) = 0$ and $d(y, z) = 0$, because that would imply that $x = y = z$ which contradicts our assumption. So we must have $d(x, y) + d(y, z) \geq 1$. Therefore $d(x, z) \leq d(x, y) + d(y, z)$ and the triangle inequality holds.

Since all the properties are satisfied, this is a legitimate metric.

To figure out what closure operator this induces, simply use the Euclidean closure operator with the new metric and "see what happens". The Euclidean closure operator is:

$$\mathbf{K}A = \{x \in X | (\forall \epsilon > 0)(\exists a \in A) d(x, a) < \epsilon\}$$

In our metric, $d(x, a)$ is always 1 or 0. If $d(x, a) = 1$ (i.e. $x$ is not in $A$), then we can take $0 < \epsilon < 1$ and $x$ is not in the closure of $A$. So the closure of $A$ contains no points that are not in $A$. On the other hand, if $d(x, a) = 0$ (i.e. $x$ is in $A$), then $d(x, a) < \epsilon$ is true for any $\epsilon > 0$. So any point in $A$ is in $\mathbf{K}A$.
Therefore, the induced closure operator is the *discrete* closure operator $\mathbf{K}A = A$.