Math S–101. Midterm 2. Solutions.

T. Judson

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1. Given a nonempty set X, show that

$$d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$

defines a metric on X.

Solution. We must verify the definition of a metric. It is clear that $d(x,y) \geq 0$ with equality if and only if $x = y$ by the definition of d. Similarly, $d(x,y) = d(y,x)$ by the definition. It remains to show that the triangle inequality holds. If $d(x,z) = 0$, then $x = z$ and $d(x,z) \leq d(x,y) + d(y,z)$. If $d(x,z) = 1$, then $x \neq z$ and either $d(x,y) = 1$ or $d(y,z) = 1$ (or both), since y must either b e distinct from x or z (or both).

2. Define the Euclidean distance between two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$ to be

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

Show that d defines a metric on $\mathbb{R}^2$.

Solution. Let $x = (x_1, x_2)$, $y = (y_1, y_2)$, and $z = (z_1, z_2)$. We wish to show that

$$\sqrt{(x_1 - z_1)^2 + (x_2 - z_2)^2} \leq \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} + \sqrt{(y_1 - z_1)^2 + (y_2 - z_2)^2}.$$

If we let $a_i = x_i - y_i$ and $b_i = y_i - z_i$ for $i = 1, 2$, then $a_i + b_i = x_i - z_i$ and our last statement is equivalent to showing that

$$\sqrt{(a_1 + b_1)^2 + (a_2 + b_2)^2} \leq \sqrt{a_1^2 + a_2^2} + \sqrt{b_1^2 + b_2^2}.$$

Since both sides are nonnegative, it is enough to show that

$$\begin{aligned} a_1^2 + 2a_1b_1 + b_1^2 + a_2^2 + 2a_2b_2 + b_2^2 &= (a_1 + b_1)^2 + (a_2 + b_2)^2 \\ &= \left(\sqrt{(a_1 + b_1)^2 + (a_2 + b_2)^2} \right)^2 \\ &\leq \left(\sqrt{a_1^2 + a_2^2} + \sqrt{b_1^2 + b_2^2} \right)^2 \\ &= a_1^2 + a_2^2 + 2\sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2} + b_1^2 + b_2^2 \end{aligned}$$

or multiplying out both sides of the inequality and canceling terms

$$a_1b_1 + a_2b_2 \le \sqrt{(a_1^2 + a_2^2)(b_1^2 + b_2^2)}.$$

Since the right-hand side of this inequality is nonnegative, it suffices to show that

$$\begin{aligned} a_1^2b_1^2 + 2a_1b_1a_2b_2 + a_2^2b_2^2 &= (a_1b_1 + a_2b_2)^2 \\ &\le (a_1^2 + a_2^2)(b_1^2 + b_2^2) \\ &= a_1^2b_1^2 + a_1^2b_2^2 + a_2^2b_1^2 + a_2^2b_2^2 \end{aligned}$$

or $2a_1b_1a_2b_2 \le a_1^2b_2^2 + a_2^2b_1^2$. However, this last statement is always true since

$$a_1^2b_2^2 - 2a_1b_1a_2b_2 + a_2^2b_1^2 = (a_1b_1 - a_2b_2)^2 \ge 0.$$

3. Consider the set $\mathbb{L} = \mathbb{Z}^+ \cup \{\infty\}$. Show that

$$d(n,m) = \begin{cases} |1/n - 1/m|, & \text{if } m, n \in \mathbb{Z}^+ \\ 1/n, & \text{if } m = \infty \end{cases}$$

defines a metric on $\mathbb{L}$.

Solution. We must verify the definition of a metric. It is clear that $d(n,m) \ge 0$ with equality if and only if $n = m$ by the definition of d. Similarly, $d(n,m) = d(m,n)$ by the definition. It remains to show that the triangle inequality holds, $d(n,m) + d(m,p) \ge d(n,p)$. If m, n, and p are in $\mathbb{Z}^+$, then

$$d(n,m) + d(m,p) = \left|\frac{1}{n} - \frac{1}{m}\right| + \left|\frac{1}{m} - \frac{1}{p}\right| \ge \left|\left(\frac{1}{n} - \frac{1}{m}\right) + \left(\frac{1}{m} - \frac{1}{p}\right)\right| = \left|\frac{1}{n} - \frac{1}{p}\right| = d(n,p).$$

Now suppose that one of the points in question is ∞. We have two cases. First suppose that $n = \infty$. Then

$$d(n,m) + d(m,p) = \left|\frac{1}{m}\right| + \left|\frac{1}{m} - \frac{1}{p}\right| \ge \left|\frac{1}{m} + \left(\frac{1}{m} - \frac{1}{p}\right)\right| = \left|\frac{2}{m} - \frac{1}{p}\right| \ge \left|\frac{1}{p}\right| = d(n,p).$$

If $m = \infty$, then

$$d(n,m) + d(m,p) = \left|\frac{1}{n}\right| + \left|\frac{1}{p}\right| \ge \left|\frac{1}{n} - \frac{1}{p}\right| = d(n,p).$$

4. Suppose that X and Y are disjoint sets. That is, $X \cap Y = \emptyset$. Suppose that $\mathbf{K}_X$ and $\mathbf{K}_Y$ are closure operators on X and Y, respectively. Define an operator $\mathbf{K}$ on $X \cup Y$ by

$$\mathbf{K}(A) = \mathbf{K}_X(A \cap X) \cup \mathbf{K}_Y(A \cap Y)$$

for each $A \subset X \cup Y$. Show that $\mathbf{K}$ is a closure operator.

Solution. We must verify each of the closure axioms.

- *Axiom C1.* We must show that $A \subset \mathbf{K}(A)$ for each $A \subset X \cup Y$. If $A \subset X \cup Y$, then

$$\mathbf{K}(A) = \mathbf{K}_X(A \cap X) \cup \mathbf{K}_Y(A \cap Y) \supset (A \cap X) \cup (A \cap Y) = A \cap (X \cup Y) = A.$$

- *Axiom C2.* We must show that $\mathbf{K}(A \cup B) = \mathbf{K}(A) \cup \mathbf{K}(B)$ for each A and B contained in $\subset X \cup Y$. If A and B are contained in $\subset X \cup Y$, then

$$\begin{aligned}
\mathbf{K}(A \cup B) &= \mathbf{K}_X((A \cup B) \cap X) \cup \mathbf{K}_Y((A \cup B) \cap Y) \\
&= \mathbf{K}_X((A \cap X) \cup (B \cap X)) \cup \mathbf{K}_Y((A \cap X) \cup (B \cap X)) \\
&= \mathbf{K}_X(A \cap X) \cup \mathbf{K}_X(B \cap X) \cup \mathbf{K}_Y(A \cap X) \cup \mathbf{K}_Y(B \cap X) \\
&= \mathbf{K}_X(A \cap X) \cup \mathbf{K}_Y(A \cap X) \cup \mathbf{K}_X(B \cap X) \cup \mathbf{K}_Y(B \cap X) \\
&= \mathbf{K}(A) \cup \mathbf{K}(B).
\end{aligned}$$

- *Axiom C3.* We must show that $\mathbf{K}(\mathbf{K}(A)) = \mathbf{K}(A)$ for each $A \subset X \cup Y$; however,

$$\begin{aligned}
\mathbf{K}(\mathbf{K}(A)) &= \mathbf{K}_X(\mathbf{K}(A) \cap X) \cup \mathbf{K}_Y(\mathbf{K}(A) \cap Y) \\
&= \mathbf{K}_X[(\mathbf{K}_X(A \cap X) \cup \mathbf{K}_Y(A \cap Y)) \cap X] \\
&\quad \cup \mathbf{K}_Y[(\mathbf{K}_X(A \cap X) \cup \mathbf{K}_Y(A \cap Y)) \cap Y] \\
&= \mathbf{K}_X[(\mathbf{K}_X(A \cap X) \cap X] \cup [\mathbf{K}_Y(A \cap Y)) \cap X] \\
&\quad \cup \mathbf{K}_Y[(\mathbf{K}_X(A \cap X) \cap Y] \cup [\mathbf{K}_Y(A \cap Y)) \cap Y] \\
&= (\mathbf{K}_X[\mathbf{K}_X(A \cap X)] \cup \emptyset) \cup \mathbf{K}_Y(\emptyset \cup \mathbf{K}_Y(A \cap Y)) \\
&= \mathbf{K}_X[\mathbf{K}_X(A \cap X)] \cup \mathbf{K}_Y[\mathbf{K}_Y(A \cap Y)] \\
&= \mathbf{K}_X(A \cap X)] \cup \mathbf{K}_Y(A \cap Y) \\
&= \mathbf{K}(A).
\end{aligned}$$

- *Axiom C4.* We must show that $\mathbf{K}(\emptyset) = \emptyset$; however,

$$\mathbf{K}(\emptyset) = \mathbf{K}_X(\emptyset \cap X) \cup \mathbf{K}_Y(\emptyset \cap Y) = \mathbf{K}_X(\emptyset) \cup \mathbf{K}_Y(\emptyset) = \emptyset \cup \emptyset = \emptyset.$$

5. Let X be a topological space with closure operator $\mathbf{K}$. If A is a connected subset of X, show that $\mathbf{K}(A)$ must be connected.

   Solution. Will will prove this result by contradiction. Suppose that $\mathbf{K}(A)$ is not connected. Then there exist nonempty sets B and C such that $\mathbf{K}(A) = B \cup C$ and

   $$\mathbf{K}(B) \cap C = B \cap \mathbf{K}(C) = \emptyset.$$

   Now let $B' = A \cap B$ and $C' = A \cap C$. We will show that B' and C' must separate A, which is a contradiction. Clearly, $A = B' \cup C'$. The sets B' and C' must be nonempty. If $B' = \emptyset$, then $A = B' \cup C' = C' = A \cap C$, which tells us that $A \subset C$. Thus, $\mathbf{K}(A) \subset \mathbf{K}(C)$. Since $\mathbf{K}(A) \supset C$, we must have equality. However,

   $$\emptyset = B \cap \mathbf{K}(C) = B \cap \mathbf{K}(A) = B$$

   contradicts the fact that $B \cap \mathbf{K}(C) = \emptyset$. Similarly, C' cannot be empty. Furthermore,

   $$\mathbf{K}(B') \cap C' \subset \mathbf{K}(B) \cap C = \emptyset.$$

   Similarly, $B' \cap \mathbf{K}(C') = \emptyset$. Thus, we have shown that A is not connected, which contradicts our hypothesis.

6. Let X be a topological space with closure operator $\mathbf{K}$. If A is connected and $A \subset B \subset \mathbf{K}(A)$, prove that B must also be connected.

Solution. Will will prove this result by contradiction. Suppose that B is not connected. Then there exist nonempty sets C and D of B such that $B = C \cup D$ and

$$\mathbf{K}(C) \cap D = C \cap \mathbf{K}(D) = \emptyset.$$

Now let $C' = C \cap A$ and $D' = D \cap A$. We will show that B' and C' must separate A, which is a contradiction. Clearly, $A = C' \cup D'$. The sets C' and D' must be nonempty. If $C' = \emptyset$, then $A = C' \cup D' = D' = A \cap D$, which tells us that $A \subset D$. Thus, $\mathbf{K}(A) \subset \mathbf{K}(D)$. Since $\mathbf{K}(A) \supset \mathbf{K}(B) \supset \mathbf{K}(D)$, we must have equality. However,

$$C = C \cap \mathbf{K}(A) = C \cap \mathbf{K}(D)$$

is nonempty, which is a contradiction. Similarly, D' cannot be empty. Furthermore,

$$\mathbf{K}(C') \cap D' \subset \mathbf{K}(C) \cap D = \emptyset.$$

Similarly, $C \cap \mathbf{K}(D) = \emptyset$. Thus, we have shown that A is not connected, which contradicts our hypothesis.

7. Let $f : X \to Y$ be a well-defined map between two sets X and Y. Define the *inverse image* of a subset $A \subset Y$ under f to be the set

$$f^{-1}(A) = \{x \in X | f(x) \in A\}.$$

Notice that f^{-1} does not necessarily define a map from Y to X.

Now let X and Y be topological spaces and $f : X \to Y$ be a well-defined map. We say that f is *continuous* if for each open set $U \subset Y$, the set $f^{-1}(U)$ is open in X.

Show that f is a continuous function if and only if the set $f^{-1}(E)$ is closed in X for each closed set E contained in Y.

Solution. Let A and B be subsets of Y. We claim that $f^{-1}(A - B) = f^{-1}(A) - f^{-1}(B)$. However,

$$\begin{aligned} x \in f^{-1}(A - B) &\iff f(x) \in A - B \\ &\iff f(x) \in A \text{ and } f(x) \notin B \\ &\iff x \in f^{-1}(A) \text{ and } x \notin f^{-1}(B) \\ &\iff x \in f^{-1}(A) - f^{-1}(B). \end{aligned}$$

Now, suppose that f is a continuous function and E is a closed subset of Y. Then $U = Y - E$ is an open subset of Y. Thus, $f^{-1}(U)$ is open in X, since f is continuous. By our remarks above

$$f^{-1}(E) = f^{-1}(Y - U) = f^{-1}(Y) - f^{-1}(U) = X - f^{-1}(U).$$

is closed.

Conversely, assume that set $f^{-1}(E)$ is closed in X for each closed set E contained in Y. If U is any open set contained in Y, then $E = Y - U$ is closed. Therefore,

$$f^{-1}(U) = f^{-1}(Y - E) = f^{-1}(Y) - f^{-1}(E) = X - f^{-1}(E).$$

must be open, and f is continuous.

8. Suppose that A and B separate X and $Y \subset X$. If $A \cap Y \neq \emptyset$ and $B \cap Y \neq \emptyset$, show that $A \cap Y$ and $B \cap Y$ separate Y.

Solution. We must show that

$$\mathbf{K}(A \cap Y) \cap (B \cap Y) = (A \cap Y) \cap \mathbf{K}(B \cap Y) = \emptyset.$$

However,

$$\mathbf{K}(A \cap Y) \cap (B \cap Y) \subset \mathbf{K}(A) \cap (B \cap Y) \subset \mathbf{K}(A) \cap B = \emptyset,$$

since A and B separate X. Similarly, $(A \cap Y) \cap \mathbf{K}(B \cap Y) = \emptyset$. Clearly, $(A \cap Y) \cup (B \cap Y) = Y$.