Math S–101. Assignment 2.
Due Tuesday, July 11, 2006

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1. Prove that $\gcd(a, c) = \gcd(b, c) = 1$ if and only if $\gcd(ab, c) = 1$ for integers a, b, and c.

2. Let $p \geq 2$. Prove that if $2^p - 1$ is prime, then p must also be prime.

3. Given the following statements,

   (a) $P \rightarrow \sim Q$
   (b) $P \leftrightarrow (P \wedge Q)$
   (c) $(P \vee Q) \wedge \sim (P \wedge Q)$
   (d) $P \rightarrow \sim P$
   (e) $(P \vee Q) \leftrightarrow (P \wedge Q)$

   match each statement with a propositionally equivalent statement below.

   (i) $P \wedge \sim P$
   (ii) $P \rightarrow Q$
   (iii) $\sim (P \wedge Q)$
   (iv) $Q \rightarrow P$
   (v) $P \leftrightarrow \sim Q$
   (vi) $\sim P$
   (vii) $P \leftrightarrow Q$
   (viii) $Q \wedge \sim P$

4. Let P, Q, and R stand for "Pigs are fish," "$2 + 2 = 4$", and "Canada is in Asia." Translate each of the following symbolic statements into reasonable-sounding English sentences. Also, determine whether each statement is true or false.

(a) $P \vee \sim Q$

(b) $Q \leftrightarrow \sim R$

(c) $\sim Q \rightarrow (R \wedge \sim P)$

(d) $P \rightarrow \sim P$

5. For each of the following statements, introduce a propositional variable for each of its atomic substatements, and then use these variables and connectives to write the most accurate symbolic translation of the original statement.

   (a) I need to go to Oxnard and Lampoc.

   (b) If a number is even and bigger than 2, then it's not prime.

   (c) If it doesn't rain in the next week, we won't have vegetables or flowers, but if it does, we'll at least have flowers.

6. **Challenge Problem**.[1] Show that

$$\sqrt[n]{a_1 a_2 \cdots a_n} \leq \frac{1}{n} \sum_{k=1}^{n} a_k.$$

[1] Let $A_n = (a_1 + \cdots + a_n)/n$ and $G_n = \sqrt[n]{a_1 \cdots a_n}$. If $n = 2$, then $a_1 - 2\sqrt{a_1 a_2} + a_2 = (\sqrt{a_1} - \sqrt{a_2})^2 \geq 0$. Therefore, $(a_1 + a_2)/2 \geq \sqrt{a_1 a_2}$. The next step is to prove the inequality for $n = 2^k$ by induction on k. Now let $2^m \geq n$. Now apply the last step to the 2^m numbers

$$a_1, \cdots, a_n, \underbrace{A_n, \cdots, A_n}_{2^m - n \text{ times}}$$

to show that $G_n \leq A_n$ for n in general