Math 101 Problem Set 1

Answers

Notes

Exercise 1.3.1

(b) and (c) are closure operators; (a) and (d) are not. Below, the axioms C1 through C4 are verified for (b) and (c), and a contradiction to one of the axioms is given for (a) and (d).

(a) $\mathbf{K}A = \{x \in \mathbb{Z} \mid (\exists a \in A)\ a < x\}$

Take, for example, $A = \{0\}$. Then $\mathbf{K}A = \{x \in \mathbb{Z} \mid x < 0\} = \{1, 2, 3, \ldots\}$. This violates Axiom C1, since $0 \notin \mathbf{K}A$.
Note that Axiom C3 is also false since $\mathbf{KK}A = \{2, 3, \ldots\} \neq \mathbf{K}A$.

(b) $\mathbf{K}A = \{x \in \mathbb{Z} \mid (\exists a \in A)\ a \leq x\}$

Let A, B be sets of integers. The reason this problem is so long is that the proofs are as explicit as I can make them. I will not usually repeat myself as much as I do below or go into this much detail.

Axiom C1 Take any $x \in A$. Then there is an $a \in A$ with $a \leq x$, namely $a = x$. Therefore by definition, $x \in \mathbf{K}A$. Since any element of A is also in $\mathbf{K}A$, we conclude $A \subset \mathbf{K}A$.

Axiom C2 Take any $x \in \mathbf{K}(A \cup B)$. Then there is an $a \in A \cup B$ such that $a \leq x$. If $a \in A$, then $a \leq x$ tells us that $x \in \mathbf{K}A$; if $a \in B$, then it tells us $x \in \mathbf{K}B$. Either way, $x \in \mathbf{K}A \cup \mathbf{K}B$, so $\mathbf{K}(A \cup B) \subset \mathbf{K}A \cup \mathbf{K}B$.

Now take any $x \in \mathbf{K}A \cup \mathbf{K}B$. If $x \in \mathbf{K}A$, then there is an $a \in A$ such that $a \leq x$; if $x \in \mathbf{K}B$, then there is an $a \in B$ such that $a \leq x$. Either way, $a \in A \cup B$ and $a \leq x$, so $x \in \mathbf{K}(A \cup B)$. Therefore, $\mathbf{K}A \cup \mathbf{K}B \subset \mathbf{K}(A \cup B)$.

From the two results above, we can conclude $\mathbf{K}A \cup \mathbf{K}B = \mathbf{K}(A \cup B)$.

Axiom C3 From Axiom C1 we know that $\mathbf{K}A \subset \mathbf{KK}A$, so once we show $\mathbf{KK}A \subset \mathbf{K}A$, these two sets must be equal.

Take any $x \in \mathbf{KK}A$. Then there is an $a \in \mathbf{K}A$ such that $a \leq x$. Since $a \in \mathbf{K}A$, there is also some $b \in A$ such that $b \leq a$. Combining these inequalities we find $b \leq x$. Because $b \in A$, we have by definition that $x \in \mathbf{K}A$, and so $\mathbf{KK}A \subset \mathbf{K}A$.

Axiom C4 If $x \in \mathbf{K}\emptyset$, then there is some $a \in \emptyset$ with $a \leq x$. Because $\emptyset$ is empty, there is never such an a, so there cannot be any x in $\mathbf{K}\emptyset$, and $\mathbf{K}\emptyset = \emptyset$.

(c) $\mathbf{K}A = \{x \in \mathbb{Z} \mid (\exists a \in A)\ (\exists k \in \mathbb{Z})\ x = ka\}$

As above, let A, B be sets of integers.

Axiom C1 Take any $x \in A$. We have $k = 1 \in \mathbb{Z}$ and $a = x \in A$ such that $x = ka$. Thus $x \in \mathbf{K}A$ and $A \subset \mathbf{K}A$.

Axiom C2 Take any $x \in \mathbf{K}A \cup \mathbf{K}B$. If $x \in \mathbf{K}A$ then there are $k \in \mathbb{Z}$ and $a \in A$ such that $x = ka$; if $x \in \mathbf{K}B$ then there are $k \in \mathbb{Z}$ and $a \in B$ such that $x = ka$. Either way, there are $k \in \mathbb{Z}$ and $a \in A \cup B$ such that $x = ka$, so $x \in \mathbf{K}(A \cup B)$, and $\mathbf{K}A \cup \mathbf{K}B \subset \mathbf{K}(A \cup B)$.

Now take any $x \in \mathbf{K}(A \cup B)$. Then there are $k \in \mathbb{Z}$ and $a \in A \cup B$ such that $x = ka$. If $a \in A$, then $x \in \mathbf{K}A$; if $a \in B$, then $x \in \mathbf{K}B$. Either way, $x \in \mathbf{K}A \cup \mathbf{K}B$, so $\mathbf{K}(A \cup B) \subset \mathbf{K}A \cup \mathbf{K}B$.

Combining the above results, $\mathbf{K}(A \cup B) = \mathbf{K}A \cup \mathbf{K}B$.

Axiom C3 As in (b), we only need to show $\mathbf{KK}A \subset \mathbf{K}A$.

Take any $x \in \mathbf{KK}A$. Then there are $k \in \mathbb{Z}$ and $a \in \mathbf{K}A$ such that $x = ka$. Since $a \in \mathbf{K}A$, there are $l \in \mathbb{Z}$ and $b \in A$ such that $a = lb$. Substituting into the previous equality, $x = k(lb) = (kl)b$.

Now since k and l are integers, $kl = j$ is also an integer. Because $x = jb$ for $j \in \mathbb{Z}$ and $b \in A$, $x \in \mathbf{K}A$, and therefore $\mathbf{KK}A \subset \mathbf{K}A$.

Axiom C4 Just like in (b), because it is never true that $a \in \emptyset$, x can never satisfy $x = ka$, so x cannot be in $\mathbf{K}\emptyset$. Therefore $\mathbf{K}\emptyset = \emptyset$.

(d) $\mathbf{K}A = \{x \in \mathbb{Z} \mid (\exists a_1 \in A)\ (\exists a_2 \in A)\ a_1 \le x \le a_2\}$

This does not satisfy Axiom C2, with sets $A = \{-1\}$ and $B = \{1\}$. $\mathbf{K}A = \{-1\}$ and $\mathbf{K}B = \{1\}$, so $\mathbf{K}A \cup \mathbf{K}B = \{-1, 1\}$. However, $A \cup B = \{-1, 1\}$, so $\mathbf{K}(A \cup B) = \{-1, 0, 1\}$. (The rest of the axioms are satisfied: feel free to prove them!)

Exercise 1.3.2

We need to show that $\mathbf{K}$ is a closure operator. As before, we verify the four axioms for a closure operator. Let A, B be subsets of L.

Axiom C1 If A is finite, then $A \subset A = \mathbf{K}A$. If A is infinite, then $A \subset A \cup \{\infty\} = \mathbf{K}A$

Axiom C2 If $A \cup B$ is finite then it has n elements, $\{a_1, a_2, \ldots, a_n\}$. Since A is a subset of this, it has the form $\{a_{n_1}, a_{n_2}, \ldots, a_{n_k}\}$, where $n_1, n_2, \ldots, n_k$ are k distinct integers between 1 and n. A then has k elements, and is finite. Similarly, we find that B is finite. Thus $\mathbf{K}A = A$ and $\mathbf{K}B = B$, so $\mathbf{K}(A \cup B) = A \cup B = \mathbf{K}A \cup \mathbf{K}B$.

If $A \cup B$ is infinite, then one of A or B must be infinite (if both were finite, A with n elements and B with m elements, then $A \cup B$ would have at most $n + m$ elements, and would be finite). If A is infinite, then $\mathbf{K}A = A \cup \{\infty\}$, so whether or not B is infinite ($\mathbf{K}B$ is either B or $B \cup \{\infty\}$), $\mathbf{K}A \cup \mathbf{K}B = A \cup B \cup \{\infty\}$. If B is infinite, $\mathbf{K}B = B \cup \{\infty\}$, and we get the same equality. In both cases, $\mathbf{K}A \cup \mathbf{K}B = (A \cup B) \cup \{\infty\} = \mathbf{K}(A \cup B)$.

Axiom C3 If A is finite, then $\mathbf{K}A = A$, so $\mathbf{K}A$ is finite, and by definition $\mathbf{KK}A = \mathbf{K}A$.

If A is infinite, then $\mathbf{K}A = A \cup \{\infty\}$, and $\mathbf{K}A$ is infinite as well (if $\mathbf{K}A$ were finite then it would have n elements, and A would have either $n - 1$ or n elements, depending on whether or not $\infty \in A$: either way, A would be finite). Thus $\mathbf{KK}A = \mathbf{K}A \cup \{\infty\} = (A \cup \{\infty\}) \cup \{\infty\} = A \cup \{\infty\} = \mathbf{K}A$.

Axiom C4 $\emptyset$ has 0 elements, so it is finite, and $\mathbf{K}\emptyset = \emptyset$.

Exercise 1.4.1

(a) Using Axiom C2 twice, $\mathbf{K}(A_1 \cup A_2 \cup A_3) = \mathbf{K}(A_1 \cup A_2) \cup \mathbf{K}A_3 = \mathbf{K}A_1 \cup \mathbf{K}A_2 \cup \mathbf{K}A_3$.

(b) Using Axiom C2 and then part (a), $\mathbf{K}(A_1 \cup A_2 \cup A_3 \cup A_4) = \mathbf{K}(A_1 \cup A_2 \cup A_3) \cup \mathbf{K}A_4 = \mathbf{K}A_1 \cup \mathbf{K}A_2 \cup \mathbf{K}A_3 \cup \mathbf{K}A_4$.

(c) The general statement from the first two parts is

$$\mathbf{K}(A_1 \cup A_2 \cup \cdots \cup A_n) = \mathbf{K}A_1 \cup \mathbf{K}A_2 \cup \cdots \cup \mathbf{K}A_n. \tag{1}$$

We've shown this for $n = 3$ and 4, and to show it for every n our best bet would be to use induction. The proof goes as follows:

- *Base Case*: The statement is trivial for $n = 1$, and for $n = 2$, $\mathbf{K}(A_1 \cup A_2) = \mathbf{K}A_1 \cup \mathbf{K}A_2$ by Axiom C2.
- *Inductive Step*: Given an $n > 2$, assume equation (1) is true. Now we need to prove (1) is true with $n + 1$ substituted for n. Using Axiom C2 and then our inductive hypothesis, $\mathbf{K}(A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1}) = \mathbf{K}(A_1 \cup A_2 \cup \cdots \cup A_n) \cup \mathbf{K}A_{n+1} = \mathbf{K}A_1 \cup \mathbf{K}A_2 \cup \cdots \cup \mathbf{K}A_n \cup \mathbf{K}A_{n+1}$. This completes the induction.

So what happens with intersections? We know from Theorem 1.10 in the notes that in general $\mathbf{K}(A_1 \cup A_2) \subset \mathbf{K}A_1 \cup \mathbf{K}A_2$, so in the analogous equation for intersections we will not have equality. The best we can hope for is

$$\mathbf{K}(A_1 \cap A_2 \cap \cdots \cap A_n) \subset \mathbf{K}A_1 \cap \mathbf{K}A_2 \cap \cdots \cap \mathbf{K}A_n, \tag{2}$$

which indeed can be proven. The proof is essentially the same as part (c) above.

Exercise 1.4.2

(b) is true. A counterexample for (a) is given by the trivial closure operator (see Notes, Example 1.3: pp. 12-13). Say $X = \{a, b\}$, and $A, B \subset X$ with $A = \{a, b\}$ and $B = \{b\}$, so $A - B = \{a\}$. With $\mathbf{K}$ as the trivial closure operator, $\mathbf{K}A = \mathbf{K}B = \mathbf{K}(A - B) = X$, so the statement (a) becomes $X \subset X - X$, or $X \subset \emptyset$, which provides a contradiction.

Before proving (b), we prove a lemma (*Hint:* Look at the hint)

Lemma 1. $C - D \subset E \Leftrightarrow C \subset D \cup E$

Proof. ($\Rightarrow$) We are given $C - D \subset E$. Take any $x \in C$. If $x \notin D$ then $x \in C - D$, so $x \in E \subset D \cup E$; if $x \in D$ then $x \in D \cup E$. Either way, we conclude $C \subset D \cup E$.

($\Leftarrow$) We are given $C \subset D \cup E$. Take any $x \in C - D$. $x \in C$, so $x \in D \cup E$, and $x \in D$ or $x \in E$. But $x \notin D$, so $x \in E$. We conclude that $C - D \subset E$. ☐

Now we can prove (b): Note that $A \subset (A - B) \cup B$ because if $x \in A$ then it is either not in B ($x \in A - B$), or *is* in B. By Theorem 1.9 in the Notes (p. 17), $\mathbf{K}A \subset \mathbf{K}[(A - B) \cup B]$. Applying Axiom C2, $\mathbf{K}A \subset \mathbf{K}(A - B) \cup \mathbf{K}B$, and so by Lemma 1, $\mathbf{K}A - \mathbf{K}B \subset \mathbf{K}(A - B)$.

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Problem 2.1.1

(a)

P	Q	$P \wedge Q$	$\sim (P \wedge Q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

(b)

P	Q	$P \vee Q$	$P \leftrightarrow (P \vee Q)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	T

(e)

P	Q	$P \wedge Q$	$Q \rightarrow (P \wedge Q)$	$P \rightarrow (Q \rightarrow (P \wedge Q))$
T	T	T	T	T
T	F	F	T	T
F	T	F	F	T
F	F	F	T	T

Problem 2.1.3

(a) $\sim (P \wedge Q) \rightarrow \sim P \wedge \sim Q$.

Let's think about this in words. The statement translates to, "If P∧Q is false, then P is false and Q is false," or, "If P and Q are not both true, then both P and Q are false." This is *neither a tautology nor a contradiction*, since if P and Q are not both true, then either (1) one of them may be true and the other false, or (2) both may be false.

(b) $\sim P \wedge \sim Q \rightarrow \sim (P \wedge Q)$

Translation: "If P and Q are both false, then P and Q are not both true." This is a *tautology* since if P and Q are false, then neither of them is true, so both can't be true.

(e) $[(P \vee Q) \vee R] \leftrightarrow [P \vee (Q \vee R)]$

The left side of this biconditional is true if any of P, Q, or R are true, and the same applies to the right side. Each side will be false only if all of P, Q, and R are false. Thus the two sides are equivalent, so this is a *tautology.*

Problem 2.1.5

(a) $P \rightarrow \sim Q$

This implication is true when P is false or when ∼Q is true (i.e. when P is false, or when P is true and ∼Q is true). This is then equivalent to $\sim P \vee \sim Q$, which is true when at least one of P and Q is false. In other words, P and Q are not both true, which is equivalent to *(iii).*

(b) $P \leftrightarrow (P \wedge Q)$

If P is false then this statement is true, or if P is true and Q is true then the statement is true as well. This condition is equivalent to $\sim P \vee Q$, which is the same as $P \rightarrow Q$ (similar to part (a)). The answer is *(ii).*

(c) $(P \vee Q) \wedge \sim (P \wedge Q)$

The left half says that at least one of P and Q is true. The right half says P and Q are not both true. If at least one of them is true, but both are not true, then one must be true and the other false. This is equivalent to *(v).*

(d) $P \rightarrow \sim P$

As before, this is equivalent to $\sim P \vee \sim P$, which is the same as ∼P, *(vi).*

(e) $(P \vee Q) \leftrightarrow (P \wedge Q)$

The left side says at least one of P and Q is true. The right says P and Q are both true. These two statements coincide when P and Q are both true or both false (if P and Q are different, then one of P and Q is true but they are not both true). Thus the correct answer is *(vii).*

Problem 2.1.6

(a) $[\sim (P \# Q)] \leftrightarrow [P \wedge \sim Q]$

The right side of the biconditional says "P is true and Q is false." This statement is *false* if P is false or if Q is true, so $P \wedge \sim Q$ is the negation of $\sim P \vee Q$. But $P \wedge \sim Q$ is also equivalent to the negation of P#Q, so P#Q is equivalent to $\sim P \vee Q$. From experience we know that $\sim P \vee Q$ is the same as $P \rightarrow Q$, so # is →.

(b) $[P \rightarrow (Q \# R)] \leftrightarrow [(P \rightarrow Q) \wedge (P \rightarrow R)]$

Reasoning this out in English is quick: If P implies Q and P implies R, then P implies Q and R. Similarly, if P implies Q and R, then P implies Q and P implies R. Thus the left and right sides imply each other if # is replaced by ∧, so ∧ is the answer.

(e) $[(P \# Q) \rightarrow R] \leftrightarrow [P \rightarrow (Q \rightarrow R)]$

We reason this out as in (b). If P being true gives that Q implies R, then P and Q both being true implies R. Similarly, if P and Q together imply R, then if P is true, Q implies R. Thus the left and right sides of the statement imply each other if # is replaced by ∧, so ∧ is the answer.

Problem 2.1.10

(a)
- O = “I need to go to Oxnard”
- L = “I need to go to Lompoc”

The statement is O $\wedge$ L

(b) Let n be a number.

- E = “n is even,” or $(\exists k \in \mathbb{Z}) n = 2k$
- B = “n is bigger than 2,” or $n > 2$
- P = “n is prime,” or $(\forall a \in \mathbb{Z}^+, a|n) a = 1$ or n

The statement is E $\wedge$ B $\rightarrow$ $\sim$P

(e)
- R = “It will rain in the next week”
- V = “We will have vegetables”
- F = “We will have flowers”

The statement is $[\sim\mathrm{R} \rightarrow \sim(\mathrm{V} \vee \mathrm{F})] \wedge [\mathrm{R} \rightarrow \mathrm{F}]$