2

We claim that for every $n \in \mathbb{N}$, the sum $1 + \frac{1}{2} + \cdots + \frac{1}{2^n}$ is less than 2. We already showed with induction that this sum is equal to

$$\frac{1(1 - \frac{1}{2^{n+1}})}{\frac{1}{2}} = 2(1 - \frac{1}{2^{n+1}}) = 2 - \frac{1}{2^n},$$

which is clearly less than 2 for all $n \in \mathbb{N}$. We claim also that 2 is the smallest such number. Let $x < 2$ such that the sum $1 + \frac{1}{2} + \cdots + \frac{1}{2^n}$ is less than x for all $n \in \mathbb{N}$. If $x < 2$, then $2 - x > 0$. Clearly for large enough $n \in \mathbb{N}$, we have $2 - x > \frac{1}{2^n}$ (for example, take n to be 10 to the number of zeros at the beginning of the decimal for $2 - x$). But then

$$\sum_{i=0}^{n} \frac{1}{2^i} = 2 - \frac{1}{2^n} < x$$

a contradiction. So 2 is indeed the smallest such number.

5

The union of two countable sets is countable. To prove this, we describe the following bijection between $\mathbb{N}$ and $A \cup B$. By hypothesis, both A and B are countable so we may list their elements as $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$. We then map all odd natural numbers $2m - 1$ to a_m and all even natural numbers $2m$ to b_m. Thus we have ordered the elements of $A \cup B$ as $a_1, b_1, a_2, b_2, \ldots$. It is clear that this map is one-to-one and onto. Thus we have a bijection and the union is also countable.

We also claim that a countable union of countable sets is countable. However, the same bijection will not work because we would never get past the first element of each set. Thus, we must find another bijection. Let $A_1, A_2, \ldots$ denote the countable sets. By hypothesis, there exists a bijection between each A_i and $\mathbb{Z}^+$. We show that the product of sets $A = A_1 \times A_2 \times \cdots$ consisting of infinite tuples of elements of each A_i is countable. The union $\cup_{i \in \mathbb{Z}^+} A_i$ is naturally a subset of A. We first map A to $\mathbb{Z}^+ \times \mathbb{Z}^+$ by sending the j-th element of A_i to (i, j). Clearly this map is a bijection. Now we use problem 4, to argue that $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable. Hence, A and therefore also $\cup_i A_i$ are as well.

6

To show that 2^A is uncountable, it suffices to show that $2^{\mathbb{Z}^+} \subset 2^A$ is uncountable. We use a proof by contradiction. Assume that the power set $2^{\mathbb{Z}^+}$ is countable. This means that we can write $2^{\mathbb{Z}^+} = \{A_1, A_2, \ldots\}$ as a list of subsets. We find an element B of $2^{\mathbb{Z}^+}$ that is not on this list. Let $B = \{k \in \mathbb{Z}^+ \mid a \notin A_k\}$. This is clearly an element of the power set. By assumption, the power set is countable, so B must equal A_n for some n. But then if $n \in B$ then $n \in A_n$ so $n \notin B$. However if $n \notin B$ then $n \notin A_n$ so $n \in B$. This is a contradiction. Thus $2^{\mathbb{Z}^+}$ is uncountable so 2^A is as well.