2. (Wolf 2.3)

d. Let P be the proposition "Pigs are blue" and Q be the proposition "Dogs are green." We are asked to find the negation of $P \leftrightarrow \sim Q$. Perhaps the easiest way to see this is with truth tables, but we instead use the tautologies given in Wolf. First $P \leftrightarrow \sim Q$ is equivalent to $(P \rightarrow \sim Q) \wedge (\sim Q \rightarrow P)$. The negation of this is $[\sim (P \rightarrow \sim Q)] \vee [\sim (\sim Q \rightarrow P)]$ which is equivalent to $(P \wedge Q) \vee (\sim Q \wedge \sim P)$. This last statement is equivalent to $P \leftrightarrow Q$. So the negation of "Pigs are blue if and only if dogs are not green" is "Pigs are blue if and only if dogs are green."

1. (Goroff 1.3)

a. If $A = \{1\}$ then $\mathbf{K}A = \{x \in \mathbb{Z} \mid x > 1\}$ which does not contain A. This violates Axiom I and thus is not a closure operator.

b. If $a \in A$ then $a \leq a$ so $a \in \mathbf{K}A$ by definition. Thus $A \subset \mathbf{K}A$ and Axiom I holds. Clearly $\mathbf{K}\emptyset = \emptyset$ so Axiom IV holds. Clearly $\mathbf{K}A \subset \mathbf{KK}A$ so we must show the converse. If $x \in \mathbf{KK}A$ then there exists some $b \in \mathbf{K}A$ such that $b \leq x$. But if $b \in \mathbf{K}A$ then there exists some $a \in A$ such that $a \leq b$. Clearly this implies that $a \leq b \leq x$ so $x \in \mathbf{K}A$ as well. So $\mathbf{K}A \subset \mathbf{KK}A$ and these sets are equal, verifying Axiom III.

The tricky axiom to check is Axiom II. If either A or B is empty, then this axiom is trivially true for $\emptyset = \mathbf{K}\emptyset$. So we assume both A and B are nonempty. If both A and B have a smallest element (called a and b respectively), then it is clear that $A = \{x \in \mathbb{Z} \mid a \leq x\}$ and likewise for B. Then it is clear that $A \cup B$ has a smallest element c equal to the minimum of a and b. So $\mathbf{K}(A \cup B) = \{x \in \mathbb{Z} \mid c \leq x\}$. Similarly, $\mathbf{K}A \cup \mathbf{K}B = \{x \in \mathbb{Z} \mid c \leq x\}$ so these sets are equal and the axiom holds. The only remaining case is when one of A and B does not contain a smallest element. Without loss of generality, we assume that A has no minimal element. We claim that in this case $\mathbf{K}A = \mathbb{Z}$. Clearly $\mathbf{K}A \subset \mathbb{Z}$. Conversely, if $x \in \mathbb{Z}$ then we know there must be some $a \in A$ such that $a \leq x$. For if this were not the case, then every element of A would be larger than x in which case A would have a minimal element. [Aside: this is one of the basic axioms about the natural numbers $\mathbb{N}$ which are equivalent to the set of integers greater than or equal to a fixed integer x.] So $x \in \mathbf{K}A$ and hence $\mathbf{K}A = \mathbb{Z}$. Now clearly $\mathbf{K}B \subset \mathbb{Z}$ so $\mathbf{K}A \cup \mathbf{K}B = \mathbb{Z}$. It is clear that if A has no smallest element then the same is true of $A \cup B$. By a similar argument, $\mathbf{K}(A \cup B) = \mathbb{Z}$ so again these sets are equal. Thus Axiom II is verified and $\mathbf{K}$ is a closure operator.

c. If $a \in A$ then $a = 1 * a$ so $a \in \mathbf{K}A$ by definition. Thus $A \subset \mathbf{K}A$ and Axiom I holds. Clearly $\mathbf{K}\emptyset = \emptyset$ so Axiom IV holds. Again $\mathbf{K}A \subset \mathbf{KK}A$ so we must show the converse. If $x \in \mathbf{KK}A$ then there exists some $b \in \mathbf{K}A$ and $m \in \mathbb{Z}$ such that $x = mb$. Similarly if $b \in \mathbf{K}A$ then there exists some $a \in A$ and $n \in \mathbb{Z}$ such that $b = na$. By substitution, this means that $x = mna$. Clearly $mn \in \mathbb{Z}$ if $m, n \in \mathbb{Z}$ so this implies that $x \in \mathbf{K}A$. Thus $\mathbf{K}A = \mathbf{KK}A$ and Axiom III holds.

Finally, let $x \in \mathbf{K}(A \cup B)$. Then there is some c in A or B and $m \in \mathbb{Z}$ such that $x = mc$. Without loss of generality, assume $c \in A$. Then clearly $x \in \mathbf{K}A$ so $x \in \mathbf{K}A \cup \mathbf{K}B$. Hence $\mathbf{K}(A \cup B) \subset \mathbf{K}A \cup \mathbf{K}B$. Now let $y \in \mathbf{K}A \cup \mathbf{K}B$. So y is in $\mathbf{K}A$ or $\mathbf{K}B$. Without loss of generality, assume $y \in \mathbf{K}A$. Then there is some $a \in A$ and $n \in \mathbb{Z}$ such that $y = na$. But clearly $a \in A \cup B$ so this means that $y \in \mathbf{K}(A \cup B)$. Thus $\mathbf{K}A \cup \mathbf{K}B \subset \mathbf{K}(A \cup B)$ and these sets are equal. So Axiom II holds and $\mathbf{K}$ is a closure operator.

d. We let $A = \{1\}$ and $B = \{3\}$. Then $\mathbf{K}(A \cup B) = \{1, 2, 3\}$ and $\mathbf{K}A \cup \mathbf{K}B = \{1, 3\}$. So $\mathbf{K}$ violates Axiom II and is not a closure operator.

2

By definition $(L, \mathbf{K})$ is a topological space if and only if $\mathbf{K}$ satisfies Axioms I, II, III, and IV on L. It is obvious from the definition that $A \subset \mathbf{K}A$ whether A is finite or infinite so Axiom I is satisfied. Also $\emptyset$ is finite so $\mathbf{K}\emptyset = \emptyset$ and Axiom IV is satisfied. If A and B are finite then there union is as well. So $\mathbf{K}(A \cup B) = A \cup B = \mathbf{K}A \cup \mathbf{K}B$. Now if either A or B is infinite then their union is infinite as well. So $\mathbf{K}(A \cup B) = A \cup B \cup \{\infty\} = \mathbf{K}A \cup \mathbf{K}B$ and Axiom II holds. Finally if A is finite, then $\mathbf{K}A = A$ is finite as well. So $\mathbf{KK}A = \mathbf{K}A$. Also if A is infinite the $\mathbf{K}A = A \cup \{\infty\}$ is also infinite. So $\mathbf{KK}A = \mathbf{K}A \cup \{\infty\} = \mathbf{K}A$ (because this set contains ∞). Thus Axiom III holds and $(L, \mathbf{K})$ is a topological space.

3

c. We conjecture that $\mathbf{K}(A_1 \cup \cdots \cup A_n) = \mathbf{K}A_1 \cup \cdots \cup \mathbf{K}A_n$ for all $n \in \mathbb{N}$. This can be proven easily with induction. The base cases were done with Axiom II and parts a and b. Assume this statement is true for some $k \in \mathbb{N}$. Then $\mathbf{K}A_1 \cup \cdots \cup \mathbf{K}A_{k+1} = (\mathbf{K}A_1 \cup \cdots \cup \mathbf{K}A_k) \cup \mathbf{K}A_{k+1} = \mathbf{K}(A_1 \cup \cdots \cup A_k) \cup \mathbf{K}A_{k+1}$ by the inductive hypothesis. By Axiom II this is equal to $\mathbf{K}((A_1 \cup \cdots \cup A_k) \cup A_{k+1}) = \mathbf{K}(A_1 \cup \cdots \cup A_{k+1})$ which is what we wanted to show. So by the Principle of Mathematical Induction we are done.