1

Let $S = \{1, 11, 111, \ldots\}$ be the set of positive integers whose digits consist entirely of 1s. It is clear that S is an infinite set. We first show that there exist two numbers in S whose *difference* is divisible by 2001. Clearly, there are exactly 2001 different remainders when we divide an integer by 2001. Because the cardinality of S is greater than 2001, by the pidgeonhole principle, there exist at least two elements of S which have the same remainder modulo 2001. Thus, their difference must have remainder 0 and be divisible by 2001.

Next we examine the form integer that is the difference of a two elements of S and divisible by 2001. It is easy to see that it has the form $11 \cdots 100 \cdots 0$. This factors as $11 \cdots 1 \times 10^n$ for some integer $n \geq 1$, in other words, as the product of an element of S times 10^n. So 2001 divides this element of S (call it a) times 10^n.

We claim that this means that 2001 divides a. To see this, we factor $2001 = 3 \times 23 \times 29$ into a product of primes. Clearly each prime must divide the product $a \times 10^n$. It is clear that for any prime p that divides a product bc of integers, either p divides b or p divides c. [Technically, this result follows from the Fundamental Theorem of Arithmetic, but you are not expected to know this.] So each of $3, 23,$and 29 must divide either a or 10^n. We can factor $10^n = 2^n \times 5^n$ into primes. If 3 divided 10^n, then it would have to divide either 2^n or 5^n by the above. Clearly, this is not the case, so 3 must divide a. A similar argument works for 23 and 29. Thus, we see that $3, 23,$ and 29 each divide a and because these numbers are prime, this implies that their product, 2001, divides a as well. As $a \in S$, our proof is complete.

2

a. Let $A = \{x \in \mathbb{R} | 0 < x < 1\}$ be the open interval $(0, 1) \subset \mathbb{R}$. We know that the closure of A should contain A because certainly points in A are close to points in A. We also know that the closure of A should contain points "infinitely close" to A. We conjecture that 0 and 1 are two such points. This is because for any possible distance, no matter how small, there is some point in A that is that close to 0 (and similarly for 1). Thus $[0, 1] \subset \mathbf{K}(A)$. To see that there are no other points in $\mathbf{K}(A)$ consider some point x outside of $[0, 1]$. Without loss of generality assume that $a > 1$. Then a cannot be infinitely close to A because it will always be at least $a - 1 > 0$ distance away from A. So $\mathbf{K}(A) = [0, 1]$.

b. Now let $B = \{x \in \mathbb{Q} | 0 < x < 1\}$. Using the same argument as above, it is clear that the points 0 and 1 are infinitely close to B and should thus be in $\mathbf{K}(B)$. Clearly B should be in $\mathbf{K}(B)$ as well. What about points in $[0, 1]$ that are not in $\mathbb{Q}$? As before, given any real number in this interval, there are points in B infinitely close to this point. This is because for any irrational number and any arbitrarily small distance, we can find a rational number within that distance of the irrational one. To do this, just take a precise enough approximation of the decimal expansion for our irrational number. We know that any finite decimal is in $\mathbb{Q}$ so this gives us a point in $B = \mathbb{Q} \cap (0, 1)$ that is within that distance of our irrational number. So $[0, 1] \cap \mathbf{K}(B)$. The same argument as above will show that no points outside of $[0, 1]$ are infinitely close to B. So $\mathbf{K}(B) = [0, 1]$.