1.2.1

a. For $n = 1, 2, 3, 4, 5, 6$, $n^2 - n + 41$ is $41, 43, 47, 53, 61$, and 71, all of which are prime.

b. If $p, a, b \in \mathbb{Z}$ such that p divides a and b, then p divides $a + b$. If we could choose n such that 41 divided $n^2 - n \neq 0$, then the result would be divisible by 41 (and greater than 41) and thus not prime. So we try $n = 41$ and see that $n^2 - n + 41 = 41^2$, which is not a prime.

1.2.2

We see that:

$$\begin{aligned}
4 &= 2 + 2\\
6 &= 3 + 3\\
8 &= 3 + 5\\
10 &= 5 + 5\\
12 &= 5 + 7\\
14 &= 7 + 7\\
16 &= 5 + 11\\
18 &= 7 + 11\\
20 &= 7 + 13\\
100 &= 47 + 53\\
102 &= 43 + 59\\
104 &= 37 + 67\\
106 &= 47 + 59\\
108 &= 37 + 71\\
110 &= 43 + 67
\end{aligned}$$

1.2.4

a. We know from class that if $m > 0$ then $m * (-1) = -m$, the additive inverse of m. By definition, any two negative numbers have the form $-m$ and $-n$ where $m, n > 0$. Then $(-m) * (-n) = (m * (-1)) * (n * (-1)) = (m * n) * ((-1) * (-1))$ by associativity and commutativity of multiplication. We know that $(-1) * (-1) = 1$, the multiplicative identity, by the same result from class. So this equals $(m * n) * 1 = (m * n)$, which we know is greater than zero.

b. Let $n \in \mathbb{R}$ be such that $n > 0$. We know that $(n - 1)^2 \geq 0$ because either $n - 1 \geq 0$, in which case this is obvious, or $n - 1 < 0$, in which case we apply the result from part a. So $n^2 - 2n + 1 \geq 0$ and $n^2 + 1 \geq 2n$. As $n > 0$, we can divide both sides by n to get $n + \frac{1}{n} \geq 2$, as desired.

1.2.5

a. $1 + 3 + 5 = 9$, $1 + 3 + 5 + 7 = 16$, and $1 + 3 + 5 + 7 + 9 = 25$.

b. The sum of the first n odd numbers equals n^2.

c. We see that $\sum_{i=1}^{9}(2i - 1) = 81 = 9^2$, $\sum_{i=1}^{10}(2i - 1) = 100 = 10^2$, $\sum_{i=1}^{11}(2i - 1) = 121 = 11^2$, and $\sum_{i=1}^{12}(2i - 1) = 144 = 12^2$.

1.2.6

a. The sum of the first n positive integer cubes is equal to the sum of the first n positive integers squared.

b. We see that $\sum_{i=1}^{4} i^3 = 100 = (\sum_{i=1}^{4} i)^2$ and $\sum_{i=1}^{5} i^3 = 175 = (\sum_{i=1}^{5} i)^2$.

1.2.10

a. 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, . . .
21, 64, 32, 16, 8, 4, 2, 1, 4, 2, 1, . . .
44, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, . . .

b. Any such sequence stabilizes in the repeating sequence $4, 2, 1, 4, 2, 1, \ldots$.

4.5.4

a. We prove this with induction on the integer n. When $n = 1$, $1[2a + (1 - 1)d]/2 = a$ so this is true. For the induction step, we assume that the statement holds for some $k \in \mathbb{N}$, i.e., that

$$a + (a + d) + \cdots + (a + (k - 1)d) = k[2a + (k - 1)d]/2$$

for some k. We want to show that this formula holds for $k + 1$. Clearly

$$\sum_{i=1}^{k+1}(a + (i - 1)d) = \left(\sum_{i=1}^{k}(a + (i - 1)d)\right) + (a + kd)$$

By our inductive hypothesis,

$$\sum_{i=1}^{k}(a + (i - 1)d) = k[2a + (k - 1)d]/2$$

so we substitute to see that

$$\begin{aligned}\sum_{i=1}^{k+1}(a + (i - 1)d) &= k[2a + (k - 1)d]/2 + (a + kd)\\ &= (2ak + k(k - 1)d + 2a + 2kd)/2\\ &= (2a(k + 1) + (k^2 + k)d)/2\\ &= (2a(k + 1) + k(k + 1)d)/2\\ &= (k + 1)[2a + ((k + 1) - 1)d]/2\end{aligned}$$

This last statement is what we wanted to show. Thus, by the Principle of Mathematical Induction, we have shown that

$$\sum_{i=1}^{n}(a + (i - 1)d) = n[2a + (n - 1)d]/2$$

for all $n \in \mathbb{N}$, and our proof is complete.

b. The sum of an arithmetic series equals the number of terms times x over 2 where x is twice the first term plus the product of the number of terms minus 1 times the difference between two consecuative terms.

4.5.8

We prove this with induction on the integer n. When $n = 1$, $a(1 - r^1)/(1 - r) = a$ so this is true. For the induction step, we assume that the statement holds for some $k \in \mathbb{N}$, i.e., that

$$\sum_{i=1}^{k} ar^{i-1} = a(1 - r^k)/(1 - r)$$

for some k. We want to show that this formula holds for $k + 1$. Clearly

$$\sum_{i=1}^{k+1} ar^{i-1} = \left(\sum_{i=1}^{k} ar^{i-1}\right) + ar^k$$

By our inductive hypothesis,

$$\sum_{i=1}^{k} ar^{i-1} = a(1 - r^k)/(1 - r)$$

so we substitute to see that

$$\begin{aligned}\sum_{i=1}^{k+1} ar^{i-1} &= a(1 - r^k)/(1 - r) + ar^k \\ &= [a(1 - r^k) + ar^k(1 - r)]/(1 - r) \\ &= [a(1 - r^k + r^k - r^{k+1})]/(1 - r) \\ &= a(1 - r^{k+1})/(1 - r)\end{aligned}$$

This last statement is what we wanted to show. Thus, by the Principle of Mathematical Induction, we have shown that

$$\sum_{i=1}^{n} ar^{i-1} = a(1 - r^n)/(1 - r)$$

for all $n \in \mathbb{N}$, and our proof is complete.

4.5.9

a. We see that $rS = ar + ar^2 + \cdots + ar^n$.

b. We see that $rS + a - ar^n = S$ so $S(1 - r) = a(1 - r^n)$ and $S = a(1 - r^n)/(1 - r)$.

4.5.14 By Theorem 4.11 (proven in class), this is equivalent to the statement that

$$\sum_{i=1}^{n} i^3 = (\frac{n(n+1)}{2})^2,$$

which we prove instead. We prove this with induction on the integer n. When $n = 1$, $(1(1+1)/2))^2 = 1 = 1^3$ so this is true. For the induction step, we assume that the statement holds for some $k \in \mathbb{N}$, i.e., that

$$\sum_{i=1}^{k} i^3 = (\frac{k(k+1)}{2})^2$$

for some k. We want to show that this formula holds for $k+1$. Clearly

$$\sum_{i=1}^{k+1} i^3 = \left(\sum_{i=1}^{k} i^3\right) + (k+1)^3$$

By our inductive hypothesis,

$$\sum_{i=1}^{k} i^3 = (\frac{k(k+1)}{2})^2$$

so we substitute to see that

$$\begin{aligned}\sum_{i=1}^{k+1} i^3 &= \frac{k(k+1)}{2})^2 + (k+1)^3 \\ &= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\ &= \frac{(k+1)^2(k^2+4k+4}{4} \\ &= \frac{(k+1)^2(k+2)^2}{4} \\ &= (\frac{(k+1)(k+2)}{2})^2\end{aligned}$$

This last statement is what we wanted to show. Thus, by the Principle of Mathematical Induction, we have shown that

$$\sum_{i=1}^{n} i^3 = (\frac{n(n+1)}{2})^2$$

for all $n \in \mathbb{N}$, and our proof is complete.