2

The classic example we've seen is in the space $\mathbb{L}$ with its usual topology. Each set $\{n\}$ for $n \in \mathbb{Z}^+$ is closed because it is finite, but the union

$$\cup_{n\in\mathbb{Z}^+}\{n\} = \mathbb{Z}^+$$

is not closed because it does not include the element ∞. For another example, consider the Euclidean topology on $\mathbb{R}$. Then each set $[1/n, 1-1/n]$ is closed but the union

$$\cup_{n\in\mathbb{Z}^+}[1/n, 1-1/n] = (0,1)$$

which is open.

1 (Wolf 3.3)

Assume $A \subset B$. We want to show that $\mathbf{K}A \subset \mathbf{K}B$. Let $x \in \mathbf{K}A$ and let $\epsilon > 0$. Because $x \in \mathbf{K}A$ there exists $a \in A$ such that $d(x,a) < \epsilon$. As $A \subset B$, $a \in B$. Thus x is within ϵ of B for arbitrary ϵ and $x \in \mathbf{K}B$. So $\mathbf{K}A \subset \mathbf{K}B$.

We no longer assume that $A \subset B$ and wish to show more generally that $\mathbf{K}(A\cap B) \subset \mathbf{K}A \cap \mathbf{K}B$. Let $x \in \mathbf{K}(A\cap B)$ and let $\epsilon > 0$. Then there exists some $c \in A\cap B$ such that $d(x,c) < \epsilon$. But clearly this $c \in A$ and $c \in B$. So x is within ϵ of A and B for each ϵ, which means that $x \in \mathbf{K}A$ and $x \in \mathbf{K}B$. So $x \in \mathbf{K}A \cap \mathbf{K}B$ and $\mathbf{K}(A\cap B) \subset \mathbf{K}A \cap \mathbf{K}B$.

3 (Wolf 3.3)

By definition $d(x,y) = 0$ when $x = y$ and is positive otherwise, so d satisfies the first axiom. Because the value of d depends only on whether $x = y$ and equality is symmetric, d is symmetric as well and satisfies the second axiom. For the third axiom, we check cases. If $x = z$ then certainly $d(x,z) \leq d(x,y) + d(y,z)$ because the right hand side is non-negative. If $x \neq z$, then clearly y cannot be equal to both x and z (by transitivity of equality). So $d(x,y) + d(y,z) \geq 1 = d(x,z)$. In either case, the third axiom is satisfied.

We claim that this metric induces the discrete closure operator (where $\mathbf{K}A = A$ for all $A \subset X$). To see this let $b \in X - A$. I claim that $b \notin \mathbf{K}A$. Because $b \notin A$ for all $a \in A$, $d(a,b) = 1$. Thus, we can find $\epsilon < 1$ for which there is no element of A within ϵ of b. Thus $b \notin \mathbf{K}A$ as desired.

4 (Wolf 3.3)

It is clear from the definition that $d(m,n)$ is zero iff $m = n$ and positive otherwise. If $m = n$ or one of m or n is infinite, then d is clearly symmetric. Otherwise $d(m,n) = |\frac{1}{m} - \frac{1}{n}| = |-(\frac{1}{m} - \frac{1}{n})| = |\frac{1}{n} - \frac{1}{m}| = d(n,m)$, so d is always symmetric. For the triangle inequality, we again check cases. If $m = n$, clearly $d(m,n) \leq d(m,q) + d(q,n)$ because the right hand side is non-negative. If $q = \infty$, then clearly $|\frac{1}{m} - \frac{1}{n}| \leq \frac{1}{m} + \frac{1}{n}$. If m or n is infinite (say WLOG $m = \infty$), then $d(m,q)+d(q,n) = \frac{1}{q}+|\frac{1}{q}-\frac{1}{n}|$, which equals $\frac{1}{n}$ if this is bigger than $\frac{1}{q}$ and is greater than $\frac{1}{q}$ otherwise. In either case this is greater than or equal to $d(m,n) = \frac{1}{n}$. If q is also infinite, we have $\frac{1}{n} \leq \frac{1}{n}$, which is clear. For the remainder of the cases we assume that m and n are finite and distinct. If q equals one of m or n (WLOG $q = m$), then $d(m,n) = |\frac{1}{m} - \frac{1}{n}| \leq |\frac{1}{m} - \frac{1}{n}| = d(m,q) + d(q,n)$. It remains only to check the cases where m, n, and q are all finite and distinct. Here we have $|mq - nq| \leq |mn - qn| + |qm - nm|$ by the usual triangle inequality $|x+y| \leq |x| + |y|$. So $\frac{q|m-n|}{mnq} \leq \frac{n|m-q|}{mnq} + \frac{m|q-n|}{mnq} \Rightarrow \frac{|m-n|}{mn} \leq \frac{|m-q|}{mq} + \frac{|q-n|}{nq} \Rightarrow |\frac{1}{m}-\frac{1}{n}| \leq |\frac{1}{m}-\frac{1}{q}|+|\frac{1}{q}-\frac{1}{n}| \Rightarrow d(m,n) \leq d(m,q)+d(q,n)$.

We claim that this metric induces the usual closure operator on $\mathbb{L}$. Let $A \subset$ Ł be finite and $x \notin A$. Then the set $\{d(a,x) \mid a \in A\}$ is finite and its smallest value c is strictly positive, because

$x \notin A$. Then if we chose $\epsilon < c$ there is no $a \in A$ within ϵ of x. So $x \notin \mathbf{K}A$ and $\mathbf{K}A = A$ for any finite set.

Now let A be an infinite set. We claim that this means that for any $N \in \mathbb{Z}^+$ there is some $n \in A \cap \mathbb{Z}^+$ such that $n \geq N$. Clearly if this were not the case, A would be finite. We'll show that $\infty \in \mathbf{K}A$, whether or not $\infty \in A$. Let $\epsilon > 0$. As ϵ is a positive real number there is some $N \in \mathbb{Z}^+$ such that $N > \frac{1}{\epsilon}$. Let $n \in A \cap \mathbb{Z}^+$ be greater than n. Then $\epsilon > \frac{1}{N} \geq \frac{1}{n} = d(n, \infty)$. So ∞ is within ϵ of A for every $\epsilon > 0$. Hence, $\infty \in \mathbf{K}A$.

It remains to show that for $x \notin A \cup \{\infty\}$, $x \notin \mathbf{K}A$. Let $B = \{a \in A | a \leq x\} \cup \{x+1\} \cup \{\infty\}$ be the subset of elements of A that are no greater than x plus the elements $x+1$ and ∞. We note that for $y \geq x+1$, $d(x, x+1) = \frac{1}{x} - \frac{1}{x+1} \geq \frac{1}{x} - \frac{1}{y} = d(x,y)$. Thus, it is clear that the minimal distance between x and an element of A is greater than or equal to the minimum distance between x and an element of B. But B is a finite set and as $x \notin A$ and $x \neq \infty$, $x \notin B$. So the minimum value c of the set $\{d(x,b) \mid b \in B\}$ is strictly positive. Thus for $\epsilon < c$, there is no $a \in A$ such that $d(x,a) < \epsilon$. So $x \notin \mathbf{K}A$ as desired. Hence, $\mathbf{K}A = A$ when A is finite and $\mathbf{K}A = A \cup \{\infty\}$ when A is infinite, as desired.