Chapter 10

The Brouwer Theorem in One Dimension

We will now prove a one-dimensional version of the Brouwer Fixed Point Theorem, which states that any continuous map from a closed interval to itself has a fixed point.

10.1 The Intermediate Value Theorem

To prove the one-dimensional Brouwer Fixed Point Theorem, we will use a very intuitive theorem about continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

Theorem 10.1 (Intermediate Value Theorem) *Suppose $a \leq b$ and $f : [a,b] \to \mathbb{R}$ is continuous. If y is between $f(a)$ and $f(b)$, then there exists $c \in [a,b]$ such that $f(c) = y$.*

We begin with a lemma. (This was Exercise 6.3.1.)

Lemma 10.2 *If S is a connected subset of $\mathbb{R}$, then S is an interval.*

Proof. Suppose S is not an interval; we will show that S is disconnected. Since S is not an interval, there are real numbers $x < y < z$ such that $x \in S$, $y \notin S$, and $z \in S$. Let $A = \{s \in S \mid s < y\}$, and let $B = \{s \in S \mid s > y\}$. I claim that (A, B) is a separation of S. First, A and B are nonempty because $x \in A$ and $z \in B$. Since $y \notin S$, $A \cup B = S$. It is clear that $A \cap B = \emptyset$. Since $A \subset (-\infty, y)$,

$$\mathbf{K}A \subset \mathbf{K}(-\infty, y) = (-\infty, y],$$

so $\mathbf{K}A \cap B = \emptyset$. Similarly, $A \cap \mathbf{K}B = \emptyset$. □

Proof of Thm. 10.1. By Theorem 6.9, $[a,b]$ is connected. By the Connected Image Theorem, $f[a,b]$ is connected. By Lemma 10.2, $f[a,b]$ is an interval.

Since $f(a)$ and $f(b)$ are in $f[a, b]$, it follows that $y \in f[a, b]$, by the definition of interval. Hence there exists $c \in [a, b]$ such that $f(c) = y$. □

This proof is rather curious in that it shows that something exists (namely, c) without suggesting how to find it. Our proof of the two-dimensional Brouwer theorem will be like this too.

We will now give a different proof of the Intermediate Value Theorem which suggests how one might find c.

Alternate proof of Thm. 10.1. Without loss of generality $f(a) < f(b)$. Let $a_1 = a$ and $b_1 = b$. Consider the average $(a_1 + b_1)/2$; if $f((a_1 + b_1)/2) = y$, we are done; if $f((a_1 + b_1)/2) < c$, let $a_2 = (a_1 + b_1)/2$ and $b_2 = b_1$; otherwise let $a_2 = a_1$ and $b_2 = (a_1 + b_1)/2$. Continuing this process, either we will find an inverse image of y, or we will obtain numbers $a_1 \leq a_2 \leq \ldots \leq b_2 \leq b_1$ such that for every n, $f(a_n) < y$, $f(b_n) > y$, and $b_{n+1} - a_{n+1} = (1/2)(b_n - a_n)$. By the Principle of Nested Closed Intervals, there exists $c \in \bigcap_{n=1}^{\infty} [a_n, b_n]$. I claim that $f(c) = y$. Suppose not. Without loss of generality, $f(c) > y$. Let $\varepsilon = f(c) - y$; since f is continuous, there exists δ such that $fB(c, \delta) \subset B(f(x), \varepsilon)$, and in particular, $f(x) > y$ for all x within distance δ of c. But for sufficiently large n, a_n is within distance δ of c, by Lemma **??**, but $f(a_n) < y$, which is a contradiction. So $f(c) = y$. □

(Notice that this proof, unlike the first, does not contain the word "connected". Do you think this is good or bad? Why?)

Suppose, for instance, that we want to find a real number x such that

$$x^5 - 4x + 2 = 0.$$

As we will see later in the course, it is impossible to write out a solution to this equation exactly in any nice way. But if we let $f(x) = x^5 - 4x + 2$, then f is a continuous function, $f(-1) = 1$, and $f(1) = -1$, so by the Intermediate Value Theorem, there exists $x \in [-1, 1]$ such that $f(x) = 0$. We can now "chase down" a solution as follows: Since $f(0) = 2 > 0$, there exists a solution in $[0, 1]$. Since $f(1/2) = 1/32 > 0$, there exists a solution in $[1/2, 1]$. Since $f(3/4) = -781/1024 < 0$, there exists a solution in $[1/2, 3/4]$. And so forth. By following this procedure, a computer can churn out a solution in base 2, one binary digit at a time.

10.2 The Brouwer theorem in one dimension

Now let us prove the one-dimensional Brouwer Fixed Point Theorem. We will prove it not just for $\mathbf{B}^1 = [-1, 1]$, but for any closed interval.

Theorem 10.3 *If $a \leq b$, then any continuous function $f : [a,b] \to [a,b]$ has a fixed point.*

Proof. Given $f : [a,b] \to [a,b]$, define

$$g(x) = f(x) - x.$$

By the Continuous Operations Theorem, g is a continuous function from $[a,b]$ to $\mathbb{R}$. Also observe that $g(a) \geq 0$ and $g(b) \leq 0$. By the Intermediate Value Theorem, there exists $x \in [a,b]$ such that $g(x) = 0$. By definition of g, this x is a fixed point of f. □

We will now outline a different and slightly less straightforward proof of this theorem. A more sophisticated version of this argument will later play a part in our proof of the Brouwer theorem in two dimensions.

Alternate Proof of Thm. 10.3. Suppose $f : [a,b] \to [a,b]$ has no fixed point. Instead of the function $g(x) = f(x) - x$ defined above, consider the function

$$r(x) = \frac{x - f(x)}{|x - f(x)|}.$$

Since $f(x) \neq x$ for all $x \in [a,b]$, the denominator is never zero, so this $r(x)$ is well-defined. By Exercise 9.2.5 (with $n = 1$ and $x_0 = 0$), absolute value is a continuous function from $\mathbb{R}$ to $\mathbb{R}$; by the Continuous Operations Theorem and the Composition Theorem, it follows that r is continuous. (How does this work?)

The only values that r ever takes are -1 and 1. Since $f(a) \neq a$, we must have $f(a) > a$, so $r(a) = -1$; similarly $r(b) = 1$. Thus

$$r[a,b] = \{-1, 1\}.$$

But this contradicts the Connected Image Theorem, because $[a,b]$ is connected while $\{-1,1\}$ is not. □

There is also a non-rigorous "picture proof" of the one-dimensional Brouwer theorem. If we draw the graph of f, we see intuitively that it must cross the line $y = x$, since it starts above the line and ends below the line, and we are not allowed to lift our pencil. The intersection of $y = f(x)$ and $y = x$ corresponds to a fixed point of f.

Exercises

1. Generalize the one-dimensional Brouwer theorem to show that if $f : [a,b] \to \mathbb{R}$ is continuous and if $f[a,b] \supset [a,b]$, then f has a fixed point.

2. Exhibit a continuous function from $(0,1)$ to $(0,1)$ with no fixed point.

Chapter 11

Topological Equivalence

"A topologist is a person who can't tell the difference between a coffee cup and a doughnut." In some sense, a coffee cup and a doughnut are the same, because if they were made out of rubber, then each one could be deformed into the other. In this chapter, we will consider the question of when two topological spaces are "the same" and when they are different. This will lead us to define an equivalence relation on the class of all topological spaces.

11.1 Injections, surjections, and bijections

Here are some properties of functions we will need.

Definition 11.1 *Suppose X and Y are sets and f is a function from X to Y. We say that f is*

- **injective** *if for any $x_1, x_2 \in X$, $x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2)$;*
- **surjective** *if for any $y \in Y$, there exists $x \in X$ such that $f(x) = y$;*
- **bijective** *if it is both injective and surjective.*

Another way to look at this is as follows: f is injective if for each $y \in Y$, the set $f^{-1}\{y\}$ contains no more than one element; f is surjective if for each $y \in Y$, $f^{-1}\{y\}$ contains at least one element; and f is bijective if for each $y \in Y$, $f^{-1}\{y\}$ contains exactly one element. An injective function is called an **injection**; a surjective function is called a **surjection**; and a bijective function is called a **bijection**.

Definition 11.2 *If $f : X \to Y$ is a bijection, the* **inverse** *of f, denoted by f^{-1} is the function from Y to X that takes an element $y \in Y$ to the unique element $x \in X$ such that $f(x) = y$.*

The notation for the inverse of a function should not be confused with that of an inverse image. If $f : X \to Y$ is a bijection and if y is an *element* of Y, then $f^{-1}(y)$ is an element of X. If A is a *subset* of Y, then the notation $f^{-1}(A)$ could apparently mean two different things: it could mean the inverse image of A under f, or the image of A under f^{-1}. But in fact, these two sets are equal (Exercise 1). For example, if $y \in Y$, then $f^{-1}(\{y\}) = \{f^{-1}(y)\}$. Inverses of functions only exist for bijections, while inverse images of sets are defined for functions that may or may not be bijective.

If f is a bijection, the inverse function f^{-1} is characterized by the fact that $f(f^{-1})(y) = y$ for all $y \in Y$ and also $f^{-1}(f(x)) = x$ for all $x \in X$. In fact, we have the following useful lemma, whose proof is Exercise 2.

Lemma 11.3 *A function $f : X \to Y$ is a bijection if and only if there exists $g : Y \to X$ such that gf is the identity on X and fg is the identity on Y. (In this case, g is the inverse of f.)*

Exercises

1. Check that if $f : X \to Y$ is a bijection and $A \subset Y$, then the inverse image of A under f and the image of A under f^{-1} are equal.

2. Suppose $f : X \to Y$ and $g : Y \to X$. Let $i : X \to X$ denote the identity function. Show that if $gf = i$ (i.e. $g(f(x)) = x$ for all $x \in X$), then f is injective and g is surjective. Deduce that Lemma 11.3 is true.

3. Suppose $f : X \to Y$.

   (a) Prove that f is injective if and only if $f^{-1}fA = A$ for all $A \subset X$.

   (b) Prove that f is surjective if and only if $ff^{-1}A = A$ for all $A \subset Y$.

   (c) Let $f(x) = x^2$ denote squaring and let $\mathbb{R}^+$ denote the nonnegative real numbers. Which of the following functions is a bijection? Which are injective? Surjective?

   - $f : \mathbb{R} \to \mathbb{R}$
   - $f : \mathbb{R}^+ \to \mathbb{R}$
   - $f : \mathbb{R} \to \mathbb{R}^+$
   - $f : \mathbb{R}^+ \to \mathbb{R}^+$

11.2 Topological equivalence

In this section we will define what it means for two topological spaces to be topologically "the same".

Let us start with an example from geometry. Recall that two triangles are congruent if the vertices of the first triangle can be paired off with the vertices of the second triangle in such a way that if x and y are two vertices of the first triangle, and if x' and y' are the corresponding vertices in the second triangle, then the distance between x and y is the same as the distance between x' and y'.

Many mathematical objects can be thought of as "sets with structure". For example, a triangle can be thought of as a set of three vertices with distances given between each pair of vertices. This distance function is a "structure" on the set of vertices. Two triangles are congruent if there is a bijection from one set to the other (that's what we really mean by "pairing off the vertices") which "respects the structure."

In the case of a topological space, the "structure" is the closure operator. Two topological spaces $(X, \mathbf{K}_X)$ and $(Y, \mathbf{K}_Y)$ are "the same" if there is a bijection $h : X \to Y$ which "respects closure." What exactly do we mean by this? The bijection h pairs up points in X with points in Y. Suppose $A \subset Y$. If we take the closure of A in Y, we want this to correspond to taking the closure in X of the points in X that correspond to A. To be precise,

$$\mathbf{K}_Y A = h\mathbf{K}_X h^{-1} A.$$

(This is an example of "conjugation", an important idea which we will discuss later.)

Definition 11.4 *Let $(X, \mathbf{K}_X)$ and $(Y, \mathbf{K}_Y)$ be topological spaces. A function $h : X \to Y$ is a* **homeomorphism** *if it is bijective and if for any $A \subset Y$,*

$$\mathbf{K}_Y A = h\mathbf{K}_X h^{-1} A.$$

If there exists a homeomorphism from $(X, \mathbf{K}_X)$ to $(Y, \mathbf{K}_Y)$, we say that $(X, \mathbf{K}_X)$ and $(Y, \mathbf{K}_Y)$ are **homeomorphic**, *or* **topologically equivalent**, *and we write $(X, \mathbf{K}_X) \approx (Y, \mathbf{K}_Y)$.*

Theorem 11.5 *$\approx$ is an equivalence relation.*

Proof. Exercise 1. □

Topological spaces that are topologically equivalent share many of the same properties. For example:

Theorem 11.6 *If $(X, \mathbf{K}_X) \approx (Y, \mathbf{K}_Y)$, then $(X, \mathbf{K}_X)$ is connected if and only if $(Y, \mathbf{K}_Y)$ is connected.*

Proof. Since $\approx$ is a symmetric relation, it suffices to show that if $(X, \mathbf{K}_X) \approx (Y, \mathbf{K}_Y)$ and $(X, \mathbf{K}_X)$ is disconnected, then $(Y, \mathbf{K}_Y)$ is disconnected. (Why?)

Suppose (A, B) is a separation of $(X, \mathbf{K}_X)$. Let $h : X \to Y$ be a homeomorphism; then I claim that $(h(A), h(B))$ is a separation of $(Y, \mathbf{K}_Y)$. Since A and B

are nonempty, $h(A)$ and $h(B)$ are nonempty. Since $A \cap B = \emptyset$ and h is injective, $h(A) \cap h(B) = \emptyset$. Since $A \cup B = X$ and h is surjective, $h(A) \cup h(B) = Y$. Since A and B are closed and h respects closure, $h(A)$ and $h(B)$ are closed. (Check this.) □

We will now give an equivalent form of the definition of homeomorphism which is useful for proving that particular pairs of spaces are homeomorphic. (This is, in fact, the standard definition.)

Theorem 11.7 (Homeomorphism Theorem) *If $(X, \mathbf{K}_X)$ and $(Y, \mathbf{K}_Y)$ are topological spaces, then $f : X \to Y$ is a homeomorphism if and only if there exists $g : Y \to X$ such that:*

(a) gf is the identity on X.

(b) fg is the identity on Y.

(c) f and g are continuous.

Equivalently (by Lemma 11.3), a homeomorphism is a continuous bijection whose inverse is continuous.

Proof. ($\Longrightarrow$) Suppose f is a homeomorphism. Since f is a bijection, $g = f^{-1}$ exists and satisfies properties (a) and (b). Now let A be any subset of X. By (a) and Definition 11.4,

$$f\mathbf{K}_X A = f\mathbf{K}_X gfA = \mathbf{K}_Y fA.$$

In particular, $f\mathbf{K}_X A \subset \mathbf{K}_Y fA$, so f is continuous. Similarly, g is continuous.

($\Longleftarrow$) Suppose f and g satisfy (a), (b), and (c). By Lemma 11.3, (a) and (b) imply that f is a bijection and $g = f^{-1}$. Let A be any subset of Y. By (b),

$$\mathbf{K}_Y A = fg\mathbf{K}_Y A.$$

Since g is continuous, $g\mathbf{K}_Y A \subset \mathbf{K}_X gA$, so by the Image Lemma,

$$fg\mathbf{K}_Y A \subset f\mathbf{K}_X gA.$$

Thus

$$\mathbf{K}_Y A \subset f\mathbf{K}_X gA.$$

Since f is continuous, and using (b),

$$f\mathbf{K}_X gA \subset \mathbf{K}_Y fgA = \mathbf{K}_Y A.$$

Thus

$$\mathbf{K}_Y A = f\mathbf{K}_X gA,$$

so f is a homeomorphism. □

Example 11.8 If $a < b$ and $c < d$, then the open intervals (a, b) and (c, d) are homeomorphic.

Proof. For $x \in (a, b)$, define

$$f(x) = c + \frac{(x-a)}{(b-a)}(d-c).$$

We leave it to the reader to check that $f : (a, b) \to (c, d)$ and that

$$g(y) = a + \frac{(y-c)}{(d-c)}(b-a)$$

is a function from (c, d) to (a, b) such that $g(f(x)) = x$ and $f(g(y)) = y$. By the Continuous Operations Theorem, f and g are continuous. By the Homeomorphism Theorem, f is a homeomorphism. □

The next example may at first seem counterintuitive.

Example 11.9 The open interval $(-1, 1)$ is homeomorphic to $\mathbb{R}$.

Proof. For $x \in (-1, 1)$, define

$$f(x) = \frac{x}{1 - |x|}.$$

Since $|x| \neq 1$ for $x \in (-1, 1)$, f is a well defined function from $(-1, 1)$ to $\mathbb{R}$. It is left as an exercise to check that

$$g(y) = \frac{y}{1 + |y|}$$

sends $\mathbb{R}$ to $(-1, 1)$ and is the inverse of f. By Exercise 9.2.5, absolute value is a continuous function from $\mathbb{R}$ to $\mathbb{R}$; continuity of f and g then follows from the Continuous Operations Theorem. □

The equivalence relation $\approx$ partitions the vast multitude of topological spaces into equivalence classes, and all of the spaces in each class are sort of alike. One can think of spaces that are homeomorphic to each other as different manifestations of the same abstract object.

Exercises

1. Prove that $\approx$ is an equivalence relation.

2. Fill in the details in the proof of Theorem 11.6.

3. (a) Fill in the details in Example 11.8. Graph the functions f and g.

(b) Do the same for Example 11.9. *Hint:* to show that $f(g(y)) = y$, consider separately the cases $y \geq 0$ and $y < 0$.

4. Suppose $a < b$ and $c < d$. Which of the following pairs of spaces are homeomorphic? Why?

   (a) $[a, b]$ and $[c, d]$.

   (b) $[a, b)$ and $[c, d)$.

   (c) $[a, b)$ and $(c, d]$.

5. Let $(L, \mathbf{K})$ be the topological space defined in Exercise 1.1.2. Show that

$$(L, \mathbf{K}) \approx \{1, 1/2, 1/3, 1/4, \ldots\} \cup \{0\}.$$

6. Show that

$$h(x) = \frac{x}{1 + \|x\|}$$

is a homeomorphism from $\mathbb{R}^n$ to $\{x \in \mathbb{R}^n \mid \|x\| < 1\}$. (Don't worry too much about continuity.)

7. The **unit sphere** in $\mathbb{R}^3$ is the set

$$S^2 \stackrel{\text{def}}{=} \{x \in \mathbb{R}^3 \mid \|x\| = 1\}.$$

Pick a point $p \in S^2$. Can you describe a function from $S^2 - \{p\}$ to $\mathbb{R}^2$ that looks like a homeomorphism? (Draw pictures.)

8. (Quotient spaces: part 1) Suppose $(X, \mathbf{K}_X)$ is a topological space, Y is a set, and $p : X \to Y$.

   (a) Show that for any subset A of Y, there is a unique smallest subset B of Y such that $A \subset B$ and $p^{-1}B$ is closed.

   (b) For $A \subset Y$, define $\mathbf{K}_Y A$ to be the set B defined in (a). Show that $\mathbf{K}_Y$ is a closure operator.

   (c) Show that with this definition of $\mathbf{K}_Y$, p is continuous.

   (d) Suppose $(W, \mathbf{K}_W)$ and $(Z, \mathbf{K}_Z)$ are topological spaces, $f : W \to X$, and $g : Y \to Z$. Show that pf is continuous if f is continuous, and gp is continuous if and only if g is continuous. *Hint:* use the Inverse Continuity Theorem.

9. (Quotient spaces: part 2) Suppose $(X, \mathbf{K})$ is a topological space and $\sim$ is an equivalence relation on X. Let $X/\sim$ denote the set of equivalence classes, and define $p : X \to X/\sim$ by taking $p(x)$ to be the equivalence class containing x. Let $\mathbf{K}_Y$ be the closure operator on $X/\sim$, as defined in part (b) of the previous exercise. The topological space $(X/\sim, \mathbf{K}_Y)$ is called a **quotient space**. Intuitively, this is the space we get when we "glue" some of the points of X together by declaring them to be equivalent.

(a) For $x, y \in \mathbb{R}$, define $x \sim y \iff x - y \in \mathbf{Z}$. Show that $\mathbb{R}/\sim$ is homeomorphic to the unit circle, $\partial\mathbf{D}^2$.

(b) Define an equivalence relation $\sim$ on $[0,1] \times [0,1]$ as follows: $(x_1, x_2) \sim (x_1', x_2')$ when $(x_1, x_2) = (x_1', x_2')$, or $x_2 = x_2'$ and $x_1, x_1' \in \{0,1\}$. Argue intuitively that $[0,1] \times [0,1]/\sim$ is homeomorphic to a cylinder in $\mathbb{R}^3$. (Draw pictures!)

(c) Define another equivalence relation $\sim$ on $[0,1]\times[0,1]$ as follows: $(x_1, x_2) \sim (x_1', x_2')$ when $(x_1, x_2) = (x_1', x_2')$, or $x_2 = x_2'$ and $x_1, x_1' \in \{0,1\}$, or $x_1 = x_1'$ and $x_2, x_2' \in \{0,1\}$. What familiar shape is $[0,1]\times[0,1]/\sim$ homeomorphic to?

(d) Define an equivalence relation on $\mathbf{D}^2$ as follows: $x \sim y$ if and only if $x = y$ or $x, y \in \partial\mathbf{D}^2$. What familiar shape is $\mathbf{D}^2/\sim$ homeomorphic to?

11.3 Topological invariants

When two spaces are homeomorphic, it is clear how to prove this: one tries to find a homeomorphism between the two spaces. (Of course, this is not always easy.) But how can we prove that two spaces are *not* homeomorphic?

Definition 11.10 *A* **topological invariant**, *or* **topological property**, *is a property of some topological spaces (for example, connectedness) such that if* $(X, \mathbf{K}_X) \approx (Y, \mathbf{K}_Y)$, *then* $(X, \mathbf{K}_X)$ *has the given property if and only if* $(Y, \mathbf{K}_Y)$ *does.*

For example, we showed in the last section that connectedness is a topological invariant. On the other hand, the statement '$6 \in X$' is not a topological property, since the closed intervals $[2,3]$ and $[5,7]$ are homeomorphic, but one contains 6 and the other does not.

The general strategy for proving that two spaces are not homeomorphic is to find a topological property that one space has and the other does not. For example, $[0,1]$ is not homeomorphic to $\{0,1\}$, because $[0,1]$ is connected while $\{0,1\}$ is not. (Of course, if two spaces are both connected, this does not necessarily imply that they are homeomorphic.)

A space $(X, \mathbf{K})$ has the **fixed point property** if every continuous function from X to itself has a fixed point. For example, the Brouwer Fixed Point Theorem states that B^n has the fixed point property.

Example 11.11 The fixed point property is a topological invariant.

Proof. Suppose $(X, \mathbf{K}_X)$ has the fixed point property, and suppose h is a homeomorphism from $(X, \mathbf{K}_X)$ to $(Y, \mathbf{K}_Y)$. We will show that $(Y, \mathbf{K}_Y)$ has the fixed point property as well. Let $g : Y \to Y$ be any continuous function. Let $f = h^{-1}gh$. By the Composition Theorem, f is a continuous function from

$(X, \mathbf{K}_X)$ to itself. Since $(X, \mathbf{K}_X)$ has the fixed point property, there is a point $x \in X$ such that $f(x) = x$, i.e., $h^{-1}(g(h(x))) = x$. By definition of h^{-1}, $g(h(x)) = h(x)$, so $h(x)$ is a fixed point of g. □

Example 11.12 The **unit circle** $S^1 = \{x \in \mathbb{R}^2 \mid \|x\| = 1\}$ is not homeomorphic to $[0, 1]$.

Proof. By the one-dimensional Brouwer theorem, $[0, 1]$ has the fixed point property. But the unit circle does not; the function $f(x) = -x$ maps the unit circle to itself, and it is continuous because it is an isometry. (See Exercise 9.2.2.) But clearly f fixes no points of the unit circle. □

The proof that the fixed point property is a topological invariant contains an important idea. Suppose $h : X \to Y$ is a homeomorphism. Then each point $x \in X$ is matched up with a point $h(x) \in Y$. If $f : X \to X$ is any function, then there is a corresponding function $g : Y \to Y$ such that if f takes x to x', then g takes $h(x)$ to $h(x')$. In other words,

$$g(h(x)) = h(f(x)),$$

or, letting $y = h(x)$,

$$g(y) = h(f(h^{-1}(y))).$$

The function $g = hfh^{-1}$ is called the **conjugation** of f by h. We can illustrate the situation with the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow h & & \downarrow h \\ Y & \xrightarrow{g} & Y \end{array}$$

If f and g are functions, and if there exists a homeomorphism h such that $g = hfh^{-1}$, then we say that f and g are **topologically conjugate**. Functions that are topologically conjugate have many of the same properties; for example, as illustrated in the previous proof, if f and g are topologically conjugate, then f has a fixed point if and only if g does.

Example 11.13 Find a continuous function from the open interval $(-1, 1)$ to itself with no fixed point.

There happens to be a simple function that works, but let's suppose we don't know what it is. We know (Example 11.9) that

$$h(x) = \frac{x}{1 + |x|}$$

Figure 11.1: A continuous function from $(-1,1)$ to itself with no fixed point.

is a homeomorphism from $\mathbb{R}$ to $(-1,1)$, and the function

$$f(x) = x + 1$$

is a continuous function from $\mathbb{R}$ to $\mathbb{R}$ with no fixed point. It follows that the conjugation

$$g = hfh^{-1}$$

is a continuous function from $(-1,1)$ to itself with no fixed point.

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{f} & \mathbb{R} \\ \downarrow h & & \downarrow h \\ (-1,1) & \xrightarrow{g} & (-1,1) \end{array}$$

For by the Composition Theorem, g is continuous; and if x is an element of $(-1,1)$ for which $g(x) = x$, then $h(f(h^{-1}(x))) = x$, so by the definition of h^{-1}, $f(h^{-1}(x)) = h^{-1}(x)$, and thus $h^{-1}(x)$ is a fixed point of f, which is a contradiction.

If we want an explicit formula for g, we plug in the formulas for f, h, and h^{-1}. From Example 11.9, we know that $h^{-1}(x) = x/(1-|x|)$. (Note that we are using different letters here.) We get

$$g(x) = h(f(h^{-1}(x))) = \frac{f(h^{-1}(x))}{1+|f(h^{-1}(x))|} = \frac{h^{-1}(x)+1}{1+|h^{-1}(x)+1|}$$

$$= \frac{\frac{x}{1-|x|}+1}{1+\left|\frac{x}{1-|x|}+1\right|} = \frac{x+1-|x|}{1-|x|+|x+1-|x||}.$$

The graph of g is shown in Figure 11.1. Notice how the curve $y = g(x)$ is always above the line $y = x$.

Incidentally, the simple function we had in mind was $g(x) = \frac{x+1}{2}$. As you can see, abstract existence proofs do not always give us the simplest answers.

The above conjugation argument generalizes to prove

Example 11.14 For any positive integer n,

$$B(0;1) = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$$

does not have the fixed point property.

At first glance, this may seem wrong; after all, the Brouwer theorem states that

$$B^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$$

does have the fixed point property, and isn't this set practically the same as $B(0;1)$? If a coffee cup is the same as a doughnut, shouldn't $B(0;1)$ be homeomorphic to B^n? Well, since the fixed point property is a topological property (and for a number of other reasons), the answer is no.

Exercises

1. Find a continuous function from $B^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ to itself with no fixed point.

2. (a) Explain why closed intervals are not homeomorphic to open intervals.
   (b) Are closed intervals homeomorphic to half-open intervals?
   (c) Are open intervals homeomorphic to half-open intervals? *Hint:* consider the topological property $(\forall x \in X)\ X - \{x\}$ is disconnected.

3. Show that the relation 'is topologically conjugate to' is an equivalence relation.

4. Let X be a set, and let $f : X \to X$ be any function. If n is a positive integer, define f^n to be the composition

$$f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n\ f's}.$$

   A point of **period** n is an element $x \in X$ such that $f^n(x) = x$.

   Prove that if $f : X \to X$ and $g : Y \to Y$ are topologically conjugate, then f has a point of period n if and only if g does.

5. Let us call a topological space $(X, \mathbf{K})$ "fragile" if there exists $x \in X$ such that $X - \{x\}$ is disconnected.
   (a) Prove that "fragility" is a topological invariant.
   (b) Show that $\mathbb{R}$ is fragile.
   (c) Show that $\mathbb{R}^2$ is not fragile, and conclude that $\mathbb{R} \not\approx \mathbb{R}^2$. *Hint:* to get started, suppose (A, B) is a separation of $\mathbb{R}^2 - \{(x_1, x_2)\}$. Use the fact that $\mathbb{R}$ is connected to show that for each $y_1 \neq x_1$, either $y_1 \times \mathbb{R} \subset A$ or $y_1 \times \mathbb{R} \subset B$.

(Using more sophisticated topological invariants, one can prove that $\mathbb{R}^m \not\approx \mathbb{R}^n$ when $m \neq n$.)

6. (fun) Draw the 26 letters of the alphabet in sans-serif capitals. You now have 26 subsets of $\mathbb{R}^2$. Which ones are homeomorphic to which? In other words, what are the equivalence classes of the set of letters of the alphabet under the equivalence relation $\approx$?

   Don't worry about being too rigorous. You will have to invent some topological properties (and show that they are topological properties); for example the topological property "there exists a continuous injection from $\mathbf{S}^1$ to X" distinguishes letters with loops, such as 'P', from letters without loops, such as 'H'. And you can argue plausibly that 'E' and 'T' are homeomorphic (without the serifs).

Part II

Background Material