Chapter 8

Continuous Functions

In this chapter we will define continuous functions, which correspond to the intuitive idea of transforming a space (like a sheet of paper) "without tearing it." We will prove the important result that continuous functions take connected sets to connected sets. To begin with, we need some ideas and notation about what functions do to sets.

8.1 Induced set functions

Recall that 2^X denotes the power set consisting of all subsets of X.

Definition 8.1 *A function $f : X \to Y$ determines an* **induced set function** $\tilde{f}$*:* $2^X \to 2^Y$ *by setting*

$$\tilde{f}(A) = \{f(x) | x \in A\}$$

for all $A \subset 2^X$.

Clearly, $\tilde{f}(\emptyset) = \emptyset$, and $\tilde{f}(\{x\}) = \{y\}$ if and only if $f(x) = y$. If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2$, for example, then $\tilde{f}(\mathbb{Z}) = \{0, 1, 4, 9, 16, \dots\}$.

In practice, it is usually unnecessary to distinguish between f and $\tilde{f}$. The difference is simply that while the first takes points in a set as arguments, the second maps subsets of the domain to subsets of the range. When we want to emphasize this difference, we refer to the original $f : X \to Y$ as a **point function**, i.e., a function that takes points to points, not sets to sets. In the future, we will usually drop the tilde from f. We call $f(A)$ the **image** of A under f, and we call $f(X)$ (that is, the image of the entire domain of f) simply the 'image of f.' Thus, we can write that f is surjective if and only if $f(X) = Y$. Here are some other useful properties of induced set functions:

Lemma 8.2 (Image Lemma) *Suppose $f : X \to Y$ and $A, B \subset X$. Then:*

(a) $f(\emptyset) = \emptyset$.

(b) $f(A \cup B) = f(A) \cup f(B)$.

(c) $A \subset B \Longrightarrow f(A) \subset f(B)$.

Proof.
(b) "$\subset$": Suppose $y \in f(A \cup B)$. Then there exists $x \in A \cup B$ such that $f(x) = y$. We know that $x \in A$ or $x \in B$. In the first case, $y \in f(A)$; in the second case, $y \in f(B)$. In both cases, $y \in f(A) \cup f(B)$. "$\supset$": Suppose $y \in f(A) \cup f(B)$. If $y \in f(A)$, then there exists $x \in A$ such that $y = f(x)$. If $y \in f(B)$, then there exists $x \in B$ such that $y = f(x)$. In both cases, $x \in A \cup B$, so $y \in f(A \cup B)$.

Parts (a) and (c) are left to the reader.

□

A function $f : X \to Y$ also induces an **inverse set-function** denoted $f^{-1} : 2^Y \to 2^X$ defined by setting

$$f^{-1}(B) = \{x \in X | f(x) \in B\}$$

for all $B \in 2^Y$. Thus, for example, f is one-to-one if and only if size$(f^{-1}(\{x\})) \leq 1$ for all $y \in Y$.

We can now state some useful properties of inverse set-functions:

Lemma 8.3 (Inverse Image Lemma) *Suppose $f : X \to Y$ and $A, B \subset Y$. Then:*

(a) $f^{-1}(\emptyset) = \emptyset$.

(b) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(c) $A \subset B \Longrightarrow f^{-1}(A) \subset f^{-1}(B)$.

(d) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

(e) $f^{-1}(A - B) = f^{-1}(A) - f^{-1}(B)$.

(f) $f^{-1}(Y) = X$.

Proof.
To prove part (b), we note that

$$\begin{aligned} x \in f^{-1}(A \cup B) &\iff f(x) \in A \cup B \\ &\iff f(x) \in A \text{ or } f(x) \in B \\ &\iff x \in f^{-1}(A) \text{ or } x \in f^{-1}(B) \\ &\iff x \in f^{-1}(A) \cup f^{-1}(B). \end{aligned}$$

The other parts are left as exercises.

□

As you can see, the inverse of a function has convenient properties that the function itself might not have, because of the way it respects set theoretic operations that the original function might not. The following lemma shows how images and inverse images relate.

Lemma 8.4 (Bouncing Lemma) *Suppose $f : X \to Y$. Then:*

(a) If $A \subset X$, then $A \subset f^{-1}fA$.

(b) If $A \subset Y$, then $ff^{-1}A \subset A$.

Proof. Exercise 6. □

This lemma confirms our intuitive notion that the inverse set function "un-does" the operations of its corresponding non-inverted function. Given $f : X \to Y$ and its inverse set function $f^{-1} : 2^Y \to 2^X$, an important question is to determine whether or not f^{-1} is the induced set-function of a point-function $g : Y \to X$: in other words, whether $f^{-1} = \tilde{g}$. If such a g exists, it follows that $g(f(x)) = x$ for all $x \in X$ and $f(g(y)) = y$ for all $y \in Y$. You can think about how to formulate necessary and sufficient conditions for this to happen, or look ahead to the Chapter on Topological Equivalence where we need the answer.

Exercises

1. Complete the proof of the Image Lemma.

2. Suppose $f : X \to Y$ and $A, B \subset X$. Show that $f(A \cap B) \subset f(A) \cap f(B)$ and $f(A - B) \supset f(A) - f(B)$. Give examples to show that the reverse inclusions do not always hold.

3. Does part (b) of the Image Lemma generalize to infinite unions?

4. Complete the proof of the Inverse Image Lemma.

5. Can you extend parts (b) and (d) of the Inverse Image Lemma to infinite unions and intersections?

6. Prove the Bouncing Lemma. Can you find examples to show that the reverse inclusions need not hold?

8.2 A Definition of continuity

Suppose $(X, \mathbf{K}_X)$ and $(Y, \mathbf{K}_Y)$ are two topological spaces. (The subscripts are used to distinguish the two closure operators.) When does a function $f : X \to Y$ transform X "without tearing it"? If $x \in X$ and $A \subset X$, and if x is really close to A, we want $f(x)$ to be really close to $f(A)$. In other words, we want $f(\mathbf{K}_X A)$ to be a subset of $\mathbf{K}_Y f(A)$. This motivates the following definition:

Definition 8.5 *Suppose $(X, \mathbf{K}_X)$ and $(Y, \mathbf{K}_Y)$ are topological spaces. A function $f : X \to Y$ is* **continuous** *if for every $A \subset X$,*

$$f\mathbf{K}_X A \subset \mathbf{K}_Y f A.$$

Example 8.6 (a) If $(X, \mathbf{K})$ is a topological space, the **identity** function $f : X \to X$, defined by $f(x) = x$, is continuous.

(b) If $(X, \mathbf{K}_X)$ and $(Y, \mathbf{K}_Y)$ are topological spaces, and if $y_0 \in Y$, then the **constant function** $g : X \to Y$ defined by $g(x) = y_0$ is continuous.

Proof. (a) For any $A \subset X$, $f\mathbf{K}A = \mathbf{K}A = \mathbf{K}fA$, so $f\mathbf{K}A \subset \mathbf{K}fA$.

(b) Suppose $A \subset X$. If $A = \emptyset$, then $g\mathbf{K}_X A$ and $\mathbf{K}_Y gA$ are both empty (by Axiom C4 and Image Lemma (a)) so $g\mathbf{K}_X A \subset \mathbf{K}_Y gA$. If $A \neq \emptyset$, then $\mathbf{K}_X A \neq \emptyset$ also (by Axiom C1), so

$$g\mathbf{K}_X A = gA = \{y_0\}.$$

By Axiom C1,

$$gA \subset \mathbf{K}_Y gA.$$

Thus $g\mathbf{K}_X A \subset \mathbf{K}_Y gA$.

□

Theorem 8.7 (Composition Theorem) *Suppose $(X, \mathbf{K}_X)$, $(Y, \mathbf{K}_Y)$, and $(Z, \mathbf{K}_Z)$ are topological spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then their composition $g \circ f : X \to Z$ defined by setting $g \circ f(x) = g(f(x))$ is also continuous. Note that it is often convenient to write gf for the composition instead of $g \circ f$.*

Proof. Suppose $A \subset X$; we need to show that $gf\mathbf{K}_X A \subset \mathbf{K}_Z gfA$, Since f is continuous, $f\mathbf{K}_X A \subset \mathbf{K}_Y fA$. By Image Lemma (c), and since g is continuous,

$$gf\mathbf{K}_X A \subset g\mathbf{K}_Y fA \subset \mathbf{K}_Z gfA.$$

□

We should mention that our definition of continuity is not the standard one, although it is equivalent. For the standard definition, see Exercise 8.3.3.

8.3 An equivalent definition of continuity

We will now prove that *a function is continuous if and only if the inverse image of every closed set is closed.* This new version of continuity, while less intuitive, is more useful in proofs, in part because inverse images, unlike images, respect all of the set-theoretic operations.

Theorem 8.8 (Inverse Continuity Theorem) *Suppose $(X, \mathbf{K}_X)$ and $(Y, \mathbf{K}_Y)$ are topological spaces. A function $f : X \to Y$ is continuous if and only if for every $A \subset Y$, if A is closed then $f^{-1}A$ is closed.*

Proof. ($\Longrightarrow$) Suppose f is continuous and A is a closed subset of Y. We must show $f^{-1}A$ is closed. We have

$$\mathbf{K}_X f^{-1}A \subset f^{-1}f\mathbf{K}_X f^{-1}A \subset f^{-1}\mathbf{K}_Y ff^{-1}A \subset f^{-1}\mathbf{K}_Y A = f^{-1}A.$$

Here the first inclusion holds by the Bouncing Lemma, the second by continuity of f and Inverse Image Lemma (c), and the third by the Bouncing Lemma, Theorem 1.9, and Inverse Image Lemma (c). Thus $\mathbf{K}_X f^{-1}A \subset f^{-1}A$. By Axiom C1, $f^{-1}A$ is closed.

($\Longleftarrow$) Suppose that the inverse image of any closed set is closed. Let $A \subset X$ be given; we must show $f\mathbf{K}_X A \subset \mathbf{K}_Y fA$. By the Boucing Lemma, $A \subset f^{-1}fA$; by Theorem 1.9 and the Image Lemma,

$$f\mathbf{K}_X A \subset f\mathbf{K}_X f^{-1}fA.$$

By Axiom C1, $fA \subset \mathbf{K}_Y fA$; thus

$$f\mathbf{K}_X f^{-1}fA \subset f\mathbf{K}_X f^{-1}\mathbf{K}_Y fA.$$

Since $\mathbf{K}_Y fA$ is closed (by Axiom C3), $f^{-1}\mathbf{K}_Y fA$ is closed by hypothesis, so

$$f\mathbf{K}_X f^{-1}\mathbf{K}_Y fA = ff^{-1}\mathbf{K}_Y fA.$$

By the Bouncing Lemma,

$$ff^{-1}\mathbf{K}_Y fA \subset \mathbf{K}_Y fA.$$

In summary,

$$f\mathbf{K}_X A \subset f\mathbf{K}_X f^{-1}fA \subset f\mathbf{K}_X f^{-1}\mathbf{K}_Y fA = ff^{-1}\mathbf{K}_Y fA \subset \mathbf{K}_Y fA.$$

Thus $f\mathbf{K}_X A \subset \mathbf{K}_Y fA$.

□

Here is another proof of the ($\Longrightarrow$) part of the above theorem, written in a different style. Suppose f is continuous and A is a closed subset of Y. We must show that $f^{-1}A$ is closed. Suppose not. Then we can choose a point $x \in X$ which is in $\mathbf{K}_X f^{-1}A$ but not in $f^{-1}A$. Since f is continuous and A is closed,

$$f(x) \in \mathbf{K}_Y ff^{-1}A \subset \mathbf{K}_Y A = A.$$

Thus $x \in f^{-1}A$, which is a contradiction.

How would you characterize the differences between these two proofs? Which proof do you prefer, and why?

Exercises

1. Use the Inverse Continuity Theorem to give an alternate proof of Example 8.6.

2. (a) Suppose $f : X \to Y$, $g : Y \to Z$, and $A \subset Z$. Show that

$$(gf)^{-1}(A) = f^{-1}(g^{-1}(A)).$$

   (b) Use the Inverse Continuity Theorem to give another proof of the Composition Theorem.

3. According to the standard definition, a function is continuous when the inverse image of every open set is open. Show that this definition is equivalent to ours. (See Exercise 2.4.2.)

4. Let $(X, \mathbf{K})$ be a discrete topological space, i.e., one where $\mathbf{K}A = A$ for all $A \subset X$. Describe the set of all continuous functions on this space. Same question for a trivial topological space.

5. Let X be a set and let f denote the identity function on X, i.e., the map that fixes every $x \in X$. Suppose that $\mathbf{K}_1$ and $\mathbf{K}_2$ are two closure operators on X. Under what circumstances is the identity function f continuous as a map from the topological space $(X, \mathbf{K}_1)$ to the topological space $(X, \mathbf{K}_2)$?

8.4 Continuous functions and connected sets

Theorem 8.9 (Connected Image Theorem 1) *Let $(X, \mathbf{K}_X)$ and $(Y, \mathbf{K}_Y)$ be topological spaces. If X is connected, if $f : X \to Y$ is continuous, and if $f(X) = Y$, then Y is connected.*

Proof. Suppose (A, B) is a separation of Y; this means that $Y = A \cup B$, $A \cap B = \emptyset$, A and B are nonempty, and A and B are closed (Proposition 4.5). I claim that $(f^{-1}A, f^{-1}B)$ is a separation of X. This contradiction will complete the proof.

By the Inverse Image Lemma,

$$f^{-1}A \cup f^{-1}B = f^{-1}(A \cup B) = f^{-1}Y = X,$$

and

$$f^{-1}A \cap f^{-1}B = f^{-1}(A \cap B) = f^{-1}\emptyset = \emptyset.$$

Since $f(X) = Y$ and A and B are nonempty, it follows that $f^{-1}A$ and $f^{-1}B$ are nonempty. By the Inverse Continuity Theorem, $f^{-1}A$ and $f^{-1}B$ are closed. □

More generally, if $f : X \to Y$ is continuous, and if A is a connected subset of X, then $f(A)$ is a connected subset of Y. We will prove this at the end of the chapter.

Exercises

1. A topological space $(X, \mathbf{K})$ has the *fixed point property* if every continuous function from X to X has a fixed point. (For example, the Brouwer fixed point theorem states that B^2 has the fixed point property.) Show that if $(X, \mathbf{K})$ has the fixed point property, then X is connected. *Hint:* Suppose (A, B) is a separation of X. Choose $a \in A$ and $b \in B$, and define

$$f(x) = \begin{cases} b & \text{if } x \in A, \\ a & \text{if } x \in B. \end{cases}$$

   Use the Inverse Continuity Theorem to show that f is a continuous function from X to X with no fixed point.

2. Show that a topological space X is disconnected if and only if there is a continuous surjective function f from X to the two point set $\{0, 1\}$.

3. Under what circumstances must the inverse image of a connected set under a continuous function be connected?

8.5 Continuity and subspaces

We now wish to extend our definition of continuity to functions that are defined on only part of a topological space. In other words, let $(X, \mathbf{K}_X)$ and $(Y, \mathbf{K}_Y)$ be topological spaces, let $W \subset X$, and let $f : W \to Y$ be a function. What should it mean for f to be continuous?

One way to answer this question is to notice that the closure operator $\mathbf{K}_X$ on X gives rise to a closure operator $\mathbf{K}_W$ on W in a natural way. If $A \subset W$, we define $\mathbf{K}_W A$ to be all points that are really close to A (according to the closure operator $\mathbf{K}_X$ on X) and contained in W. To be precise:

Definition 8.10 *Let $(X, \mathbf{K}_X)$ be a topological space and let $W \subset X$. The* **relative closure operator** $\mathbf{K}_W$ *on W is defined by*

$$\mathbf{K}_W A = W \cap \mathbf{K}_X A$$

for all $A \subset W$. We also call $\mathbf{K}_W$ the closure operator on W **induced** *by $\mathbf{K}_X$.*

Example 8.11 If $W \subset \mathbb{R}^m$, the closure operator on W induced by the standard closure operator is

$$\mathbf{K}_W A = \{x \in W \mid (\forall \varepsilon > 0)\ (\exists a \in A)\ d(x, a) < \varepsilon\}.$$

We will say that a function $f : W \to Y$ is continuous if it is continuous when regarded as a function from the topological space $(W, \mathbf{K}_W)$ to the topological space $(Y, \mathbf{K}_Y)$: that is to say, if

$$f \mathbf{K}_W A \subset \mathbf{K}_Y f A$$

for every $A \subset W$. Putting in the definition of $\mathbf{K}_W$, we see that this is equivalent to

$$f(W \cap \mathbf{K}_X A) \subset \mathbf{K}_Y f A.$$

In other words, if x is really close to A and if $f(x)$ is defined (i.e. if $x \in W$), then $f(x)$ is really close to $f(A)$.

At some point we ought to check that $\mathbf{K}_W$ is a closure operator. We will leave this proof to the reader, as it provides a useful review of Chapter 1.

We can now give a precise statement of the Brouwer fixed point theorem. While our focus is on the unit disk, the Brouwer theorem, suitably stated, works in any number of dimensions. Define the **closed unit ball** to be

$$B^n \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid d(x, 0) \leq 1\}.$$

Theorem 8.12 (Brouwer Fixed Point Theorem) *If $f : B^n \to B^n$ is continuous, then f has a fixed point.*

Figure 8.1: The graph of a function and its restriction to $[0, 1]$.

In Chapter **??**, we will prove this for $n = 1$; in Chapter **??**, we will prove it for $n = 2$. You can learn a proof for $n > 2$ in a more advanced topology course.

It behooves us to explain how this mathematical statement bears upon the physical situation we described in the Introduction. If we define a function $f : B^2 \to B^2$ by crumpling up a piece of paper in $\mathbb{R}^3$ and then projecting down to B^2, how do the laws of physics predict that f will be continuous, so that our theorem applies? This will become clear in the next chapter. (See Exercise 9.2.4.)

Exercises

1. Prove that $\mathbf{K}_W$, as given in Definition 8.10, is a closure operator, using the fact that $\mathbf{K}_X$ is a closure operator. *Hint:* for Axiom C4, use Theorem 1.10.

8.6 More subspaces*

Definition 8.13 *Let X and Y be any sets, let $f : X \to Y$ be a function, and let $W \subset X$. The* **restriction** *of f to W, which we denote by $f_{|W}$, is a function from W to Y defined by*

$$(f_{|W})(x) \stackrel{\text{def}}{=} f(x)$$

for every $x \in W$.

For example, if $f : \mathbb{R} \to \mathbb{R}$ is the function graphed in Figure 8.1(a), the restriction of f to $[0, 1]$ is graphed in Figure 8.1(b).

Lemma 8.14 *Suppose $(X, \mathbf{K}_X)$ and $(Y, \mathbf{K}_Y)$ are topological spaces, let $W \subset X$, and suppose $f : X \to Y$ is continuous. Then $f_{|W}$ is continuous.*

Proof. Let A be any subset of W; we need to show that $f_{|W}\mathbf{K}_W A \subset \mathbf{K}_Y f_{|W} A$. By definition,

$$f_{|W}\mathbf{K}_W A = f_{|W}(W \cap \mathbf{K}_X A) = f(W \cap \mathbf{K}_X A).$$

By the Image Lemma and the assumption that f is continuous,

$$f(W \cap \mathbf{K}_X A) \subset f(\mathbf{K}_X A) \subset \mathbf{K}_Y f A.$$

□

The next lemma concerns functions from X to Y whose images do not cover all of Y.

Lemma 8.15 *Let $(X, \mathbf{K}_X)$ and $(Y, \mathbf{K}_Y)$ be topological spaces, and let $Z \subset Y$. Suppose $f : X \to Y$ is a function such that $f(X) \subset Z$. Let $\mathbf{K}_Z$ be the relative closure operator on Z. Then f is continuous when regarded as a function from $(X, \mathbf{K}_X)$ to $(Y, \mathbf{K}_Y)$ if and only if f is continous when regarded as a function from $(X, \mathbf{K}_X)$ to $(Z, \mathbf{K}_Z)$.*

Proof. Exercise 1.

□

Lemma 8.16 *Suppose $(X, \mathbf{K}_X)$ is a topological space, suppose $W \subset X$, and suppose $V \subset W$. Then V is connected in $(W, \mathbf{K}_W)$ if and only if V is connected in $(X, \mathbf{K}_X)$.*

Proof. Exercise 2.

□

We can now generalize Theorem 8.9.

Theorem 8.17 (Connected Image Theorem 2) *Let $(X, \mathbf{K}_X)$ and $(Y, \mathbf{K}_Y)$ be topological spaces, and let $f : X \to Y$ be continuous, and suppose A is a connected subset of X. Then $f(A)$ is a connected subset of Y.*

Proof. By Lemmas 8.14 and 8.15, the function

$$f_{|A} : A \to f(A)$$

is continuous, where A and $f(A)$ are given the relative closure operators induced by $\mathbf{K}_X$ and $\mathbf{K}_Y$. By the Connected Image Theorem 1, $f_{|A}(A)$ is connected in $f(A)$. Evidently $f_{|A}(A) = f(A)$, and by Lemma 8.16, $f(A)$ is connected in Y.

□

Exercises

1. Prove Lemma 8.15. (Show that for every $A \subset X$, $f\mathbf{K}_X A \subset \mathbf{K}_Y fA$ if and only if $f\mathbf{K}_X A \subset \mathbf{K}_Z fA$.)

2. Prove Lemma 8.16.

Chapter 9

Limits and Continuity in $\mathbb{R}^n$

In this chapter we will figure out what our definition of continuity means for the special case of functions from one metric space to another with the induced topologies. In particular, we will then be able to prove that lots of functions from $\mathbb{R}^m$ to $\mathbb{R}^n$ are continuous. To begin with, we define what it means for a function to be continuous at a given point. This localized version of continuity can be used to give a useful and unified, if somewhat nonstandard, account of limits.

9.1 Local continuity and limits

Let $(X, \mathbf{K}_X)$ and $(Y, \mathbf{K}_Y)$ denote topological spaces. Recall that we defined a function $f : X \to Y$ to be continuous if $f(\mathbf{K}_X A) \subset \mathbf{K}_Y f(A)$ holds for all $A \subset X$. In this and in the equivalent characterizations of continuity we have studied so far, a function is either continuous or not. If not, then there must exist a set $A \subset X$ and a point $x \in \mathbf{K}_X A$ such that $f(x) \notin \mathbf{K}_Y f(A)$. It makes sense in this case to say that f is not continuous at that x. This motivates the following definition of local continuity.

Definition 9.1 *Let $(X, \mathbf{K}_X)$ and $(Y, \mathbf{K}_Y)$ denote topological spaces. For fixed $x \in X$, we say that $f : X \to Y$ is* **continuous at** *x if, for all $A \subset X$, we have that $x \in \mathbf{K}_X A$ implies $f(x) \in \mathbf{K}_Y f(A)$.*

It is straightforward to verify that f is continous on X if and only if f is continous at each and every $x \in X$. On the other hand, a function can be continuous at some points and not at others. The whole idea of a limit can be viewed as trying to extend a given function so that it is continuous at the point where the limit is taken.

Definition 9.2 *Let $(X, \mathbf{K}_X)$ and $(Y, \mathbf{K}_Y)$ denote topological spaces. Fix $a \in X$ and suppose $f : (X - \{a\}) \to Y$ is given. We write $\lim_{x \to a} f(x) = b$ and say that* *the limit of f as x approaches a exists and equals b if and only if the function $g : X \to Y$ determined by setting*

$$g(x) = \begin{cases} f(x) & \text{if } x \neq a \\ b & \text{if } x = a \end{cases}$$

is continuous at a. In this case, we can also say that $f(x)$ converges to b as x goes to b and write $f(x) \to b$ as $x \to a$.

For example, a function $f : \mathbb{Z}^+ \to Y$ is usually called a sequence in Y. What does it mean for this sequence to converge to $b \in Y$? Take $X = \mathbb{Z}^+ \cup \{\infty\}$ and $a = \infty$ in the above definition. Now X becomes the topological space we have called $\mathbb{L}$ when we declare all finite sets to be closed and all infinite sets to contain ∞ in their closure. The question is whether $g : X \to Y$ with $g(n) = f(n)$ for $n \in \mathbb{Z}^+$ and $g(\infty) = b$ extends f to a function on X that is continuous at ∞. This means precisely that for every infinite $A \subset \mathbb{Z}^+$, we must have $b \in \mathbf{K}_Y f(A)$, in which case we write $\lim_{n\to\infty} f(n) = b$. That such a characterization of the limit is equivalent to the usual definition will become more apparent after we study continuity on metric spaces in the next section.

9.2 Delta-epsilon continuity

We will start by giving an alternate definition of continuity for the case when (X, d_X) and (Y, d_Y) are metric spaces topologized in the standard way. Intuitively, a function $f : X \to Y$ is continuous if for every $x \in X$, points close to x are mapped to points close to $f(x)$. How close to $f(x)$ must we get? Let us say that for *every* $\varepsilon > 0$, points sufficiently close to x are mapped to within distance ε of $f(x)$. How close to x must we get? Let us say that for some $\delta > 0$, all points within distance δ of x are mapped to points within distance ε of $f(x)$. In other words,

$$(\forall x \in X)\ (\forall \varepsilon > 0)\ (\exists \delta > 0)\ fB(x;\delta) \subset B(f(x);\varepsilon).$$

This is called the **delta-epsilon** characterization of continuity. Notice that it is a pointwise characterization in the sense that it provides a local condition, namely

$$(\forall \varepsilon > 0)\ (\exists \delta > 0)\ fB(x;\delta) \subset B(f(x);\varepsilon).$$

to be satisfied at each fixed $x \in X$. Notice also that the open ball $B(x;\delta)$ in X is determined using the metric d_X while the open ball $B(f(x);\varepsilon)$ in Y is determined using the metric d_Y. Happily, the delta-epsilon characterization of continuity is equivalent to Definition 8.5 in metric spaces, as we will now show. Moreover, when you are given a function on Euclidean spaces and you need to prove that it is continuous, the delta-epsilon form of continuity is usually easier to use than Definition 8.5.

Theorem 9.3 (Delta-Epsilon Continuity Theorem) *Let (X, d_X) and (Y, d_Y) denote metric spaces. With respect to the closure operators on each induced by the metrics, a function $f : X \to Y$ is continuous (according to Definition 8.5) if and only if*

$$(\forall x \in X)\ (\forall \varepsilon > 0)\ (\exists \delta > 0)\ fB(x;\delta) \subset B(f(x);\varepsilon).$$

Proof. ($\Longrightarrow$) Suppose $f\mathbf{K}A \subset \mathbf{K}fA$ for all $A \subset \mathbb{R}^m$. Let $x \in \mathbb{R}^m$ and $\varepsilon > 0$ be given. We need to find $\delta > 0$ such that $fB(x;\delta) \subset B(f(x);\varepsilon)$. Suppose no such δ exists. Then

$$(\forall \delta > 0)\ (\exists a \in B(x;\delta))\ f(a) \notin B(f(x);\varepsilon).$$

Let

$$A = \{a \in \mathbb{R}^m \mid f(a) \notin B(f(x);\varepsilon)\}.$$

Then

$$(\forall \delta > 0)\ B(x;\delta) \cap A \neq \emptyset.$$

This means that $x \in \mathbf{K}A$. By assumption, $f(x) \in \mathbf{K}fA$. But this is a contradiction, because by definition of A,

$$B(f(x);\varepsilon) \cap fA = \emptyset.$$

($\Longleftarrow$) Suppose

$$(\forall x \in \mathbb{R}^m)\ (\forall \varepsilon > 0)\ (\exists \delta > 0)\ fB(x;\delta) \subset B(f(x);\varepsilon).$$

Let $A \subset \mathbb{R}^m$ be given; we must show that $f\mathbf{K}A \subset \mathbf{K}fA$. Suppose $y \in f\mathbf{K}A$; then $y = f(x)$ for some $x \in \mathbf{K}A$. We wish to show that $f(x) \in \mathbf{K}fA$, i.e. $(\forall \varepsilon > 0)\ B(f(x);\varepsilon) \cap fA \neq \emptyset$. Let $\varepsilon > 0$ be given. By assumption, we can choose $\delta > 0$ such that $fB(x;\delta) \subset B(f(x);\varepsilon)$. Since $x \in \mathbf{K}A$, $B(x;\delta) \cap A \neq \emptyset$. This implies $fB(x;\delta) \cap fA \neq \emptyset$. (Why?) Since $fB(x;\delta) \subset B(f(x);\varepsilon)$, it follows that $B(f(x);\varepsilon) \cap fA \neq \emptyset$. □

The delta-epsilon version of continuity can be restated for Euclidean spaces as follows:

$$(\forall x \in \mathbb{R}^m)\ (\forall \varepsilon > 0)\ (\exists \delta > 0)\ (\forall y \in \mathbb{R}^m)\ d(x,y) < \delta \Longrightarrow d(f(x), f(y)) < \epsilon.$$

Historically, the delta-epsilon definition of continuity was in use long before more simple, general, and abstract definitions such as Definition 8.5 were discovered.

Example 9.4 Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x$ is continuous.

Given $x \in \mathbb{R}$ and $\varepsilon > 0$, we need to find $\delta > 0$ such that

$$|x - y| < \delta \Longrightarrow |2x - 2y| < \varepsilon.$$

Recall that for real numbers a and b, $|ab| = |a| \cdot |b|$. We can now put $\delta = \varepsilon/2$; if $|x - y| < \varepsilon/2$, then

$$|2x - 2y| = 2|x - y| < 2(\varepsilon/2) = \varepsilon. \qquad \square$$

Intuitively, a function from $\mathbb{R}$ to $\mathbb{R}$ is continuous if you can draw its graph without lifting your pencil (except perhaps to sharpen it!).

Just for fun, we will now give another proof of the Delta-Epsilon Continuity Theorem. This time we will prove that delta-epsilon continuity is equivalent to the statement that the inverse image of every closed set is closed.

First, assume that $f : \mathbb{R}^m \to \mathbb{R}^n$ is delta-epsilon continuous. We wish to show that the inverse image of every closed set is closed. So let $A \subset \mathbb{R}^n$ be a closed set. We wish to show that $f^{-1}A$ is closed. By Axiom C1, it is enough to show that $\mathbf{K}f^{-1}A \subset f^{-1}A$. Suppose $x \in \mathbb{R}^m$ and $x \notin f^{-1}A$; we wish to show that $x \notin \mathbf{K}f^{-1}A$. Since $f(x) \notin A$ and A is closed, $f(x) \notin \mathbf{K}A$. Hence there exists $\varepsilon > 0$ such that

$$B(f(x);\varepsilon) \cap A = \emptyset.$$

By delta-epsilon continuity, there exists $\delta > 0$ such that

$$fB(x;\delta) \subset B(f(x);\varepsilon),$$

which by the above implies that

$$fB(x;\delta) \cap A = \emptyset$$

which is equivalent to

$$B(x;\delta) \cap f^{-1}A = \emptyset.$$

(Why?) This means that $x \notin \mathbf{K}f^{-1}A$, which is what we wanted.

Next, assume that the inverse image of every closed set is closed; we wish to show that f is delta-epsilon continuous. Let $x \in \mathbb{R}^m$ and $\varepsilon > 0$ be given; we wish to find $\delta > 0$ such that

$$fB(x;\delta) \subset B(f(x);\varepsilon).$$

Let

$$A = \mathbb{R}^n - B(f(x);\varepsilon).$$

This is closed (Exercise 6), so $f^{-1}A$ is closed. Since $x \notin f^{-1}A$, this means that there exists $\delta > 0$ such that

$$B(x;\delta) \cap f^{-1}A = \emptyset.$$

This is equivalent to

$$fB(x;\delta) \cap A = \emptyset$$

which in turn is equivalent to

$$fB(x;\delta) \subset \mathbb{R}^n - A = B(f(x);\varepsilon).$$

Exercises

1. In each of the following examples, mathematician Alpha has a function $f : \mathbb{R} \to \mathbb{R}$ which she claims is continuous. Mathematician Beta is not so sure; he gives an $x \in \mathbb{R}$ and an $\varepsilon > 0$, and he challenges Alpha to find a $\delta > 0$ such that

$$(\forall y \in \mathbb{R})\ |x-y| < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon.$$

   Can Alpha refute the challenge by finding a δ that works?

   (a) $f(x) = x$; $x = 3$; $\varepsilon = 1/17$.

   (b) $f(x) = x^2 + x + 1$; $x = 2$; $\varepsilon = 6$.

   (c) $f(x) = \sin x$; $x = \pi/2$; $\varepsilon = 1/2$.

   (d) $f(x) = \sqrt{|x|}$; $x = 4$; $\varepsilon = 1/3$.

2. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is a function such that for all $x, y \in \mathbb{R}^n$,

$$d(f(x), f(y)) = d(x,y).$$

   (Such a function is called an **isometry**.) Show that f is continuous.

3. Define $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

   Give two proofs that f is not continuous, one using Definition 8.5 and the other using delta-epsilon continuity. (f is said to have a **jump discontinuity** at 0.) In terms of local continuity, show that f is continuous at every point except 0.

4. (a) Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be the "projection" function defined by

$$f((x_1, x_2, x_3)) = (x_1, x_2).$$

   Show that f is continuous. *Hint:* first show that $d(f(x), f(y)) \leq d(x,y)$.

   (b) Give a plausible argument why the function $c : B^2 \to \mathbb{R}^3$ obtained by crumpling up a piece of paper is continuous.

   (c) Explain why our mathematical Brouwer theorem implies the statement in the Introduction about crumpling up pieces of paper.

5. Fix a point $x_0 \in \mathbb{R}^n$. Show that the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by
$$f(x) = d(x, x_0)$$
is continuous.

6. Suppose $a \in \mathbb{R}^n$ and $\varepsilon > 0$. Show that $\mathbb{R}^n - B(a;\varepsilon)$ is closed.

7. Suppose (X, d) a metric space topologized in the standard way and let $B \subset X$. Show that $b \in \mathbf{K}B$ if and only if there is a sequence $f : \mathbb{Z}^+ \to B$ such that $\lim_{n\to\infty} = b$.

8. Show that if $f : \mathbb{R} \to \mathbb{R}$ is increasing and surjective, then f is continuous. (f is **surjective** if for every $y \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $f(x) = y$. f is **increasing** if $f(x) < f(y)$ whenever $x < y$.)

9. Proof or counterexample: According to the definition we gave, must the limit of a sequence be unique? Does it matter whether or not there is a metric around?

10. The order of quantification matters. For $f : X \to Y$ a function from one metric space to another, we say that f is **uniformly continuous** on X if
$$(\forall \varepsilon > 0)\ (\exists \delta > 0)\ (\forall x \in X)\ fB(x;\delta) \subset B(f(x);\varepsilon).$$
Are uniformly continuous functions necessarily continuous? Must a continuous function be uniformly continuous? Give examples and arguments. How would you describe a function $f : \mathbb{R} \to \mathbb{R}$ that satisfies
$$(\forall x \in \mathbb{R})\ (\exists \delta > 0)\ (\forall \varepsilon > 0)\ fB(x;\delta) \subset B(f(x);\varepsilon)?$$
What can you say if X is a space other than the reals?

11. Consider the topological space $\mathbb{L} = \mathbb{Z}^+ \cup \{\infty\}$. The metric on $\mathbb{L}$ given by
$$d(n, m) = \begin{cases} |1/n - 1/m| & \text{if } m, n \in \mathbb{Z}^+ \\ 1/n & \text{if } m = \infty\ . \end{cases}$$
determines the standard closure we have studied on this space. Use it to show that, if $f : \mathbb{Z}^+ \to Y$ for some metric space (Y, d_Y), then $\lim_{n\to\infty} f(n) = b$ in Y as we have defined this is equivalent to the usual definition, namely that for all $\varepsilon > 0$ there is an $N \in \mathbb{Z}^+$ such that $n \geq N$ implies $d_Y(f(n), b) < \varepsilon$.

12. Referring to the previous two exercises, show that when $f : \mathbb{Z}^+ \to Y$ for some metric space (Y, d_Y), we have that f is uniformly continuous on $\mathbb{Z}^+$ if and only if for all $\varepsilon > 0$ there is an $N \in \mathbb{Z}^+$ such that $m, n > N$ implies $d_Y(f(m), f(n)) < \varepsilon$. We call such an f a **Cauchy sequence** in Y. Prove that a convergent sequence is Cauchy. Assuming $Y = \mathbb{R}$, prove that a Cauchy sequence converges. What if $Y \neq \mathbb{R}$? What property of the reals did you use when showing that its Cauchy sequence necessarily converge? A metric space on which every Cauchy sequence converges is said to satisfy the Cauchy Convergence Criteria (CCC). Care to make a conjecture?

9.3 Making new continuous functions out of old

We will now show some of the ways that continuous functions may be combined to create new ones. We will see that many functions are continuous, including all polynomials.

Definition 9.5 *Let X be any set, suppose f and g are functions from X to $\mathbb{R}$, and let c be any real number. The functions $c \cdot f$, $f + g$, and $f \cdot g$ are defined as follows:*

$$\begin{aligned}(c \cdot f)(x) &\stackrel{\text{def}}{=} c \cdot f(x),\\ (f+g)(x) &\stackrel{\text{def}}{=} f(x) + g(x),\\ (f \cdot g)(x) &\stackrel{\text{def}}{=} f(x) \cdot g(x).\end{aligned}$$

If $f(x) \neq 0$ for all $x \in X$, then $1/f$ is defined by

$$(1/f)(x) \stackrel{\text{def}}{=} \frac{1}{f(x)}.$$

The proof of the following theorem is a little messy, but it is useful to know how these kinds of things can be done. In particular, you should understand the techniques used to prove parts (a) and (b).

Theorem 9.6 (Continuous Operations Theorem) *If $f, g : \mathbb{R}^n \to \mathbb{R}$ are continuous, and if $c \in \mathbb{R}$, then:*

(a) $c \cdot f$ *is continuous.*

(b) $f + g$ *is continuous.*

(c) $f \cdot g$ *is continuous.*

(d) *If $f(x) \neq 0$ for all $x \in \mathbb{R}^n$, then $1/f$ is continuous.*

Proof. (a) Let $x \in \mathbb{R}^n$ and $\varepsilon > 0$ be given; we must find $\delta > 0$ such that for all $y \in \mathbb{R}^n$,

$$d(x,y) < \delta \Longrightarrow |cf(x) - cf(y)| < \varepsilon.$$

Since f is continuous, we can choose $\delta > 0$ such that

$$d(x,y) < \delta \Longrightarrow |f(x) - f(y)| < \frac{\varepsilon}{1+|c|}.$$

Then

$$d(x,y) < \delta \Longrightarrow |cf(x) - cf(y)| = |c| \cdot |f(x) - f(y)| < \frac{|c|\varepsilon}{1+|c|} < \varepsilon.$$

(We use $\varepsilon/(1+|c|)$ instead of just $\varepsilon/|c|$ in order to avoid dividing by zero.)

(b) Let $x \in \mathbb{R}^n$ and $\varepsilon > 0$ be given. Since f is continuous, we can choose $\delta_1 > 0$ such that

$$d(x,y) < \delta_1 \Longrightarrow |f(x) - f(y)| < \varepsilon/2.$$

Since g is continuous, we can choose $\delta_2 > 0$ such that

$$d(x,y) < \delta_2 \Longrightarrow |g(x) - g(y)| < \varepsilon/2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. By the triangle inequality,

$$d(x,y) < \delta \Longrightarrow |(f(x)+g(x)) - (f(y)+g(y))| \leq$$
$$\leq |f(x) - f(y)| + |g(x) - g(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

(c) Let $x \in \mathbb{R}^n$ and $\varepsilon > 0$ be given. Since f is continuous, we can choose $\delta_1 > 0$ such that

$$d(x,y) < \delta_1 \Longrightarrow |f(x) - f(y)| < \min\left\{\sqrt{\varepsilon/3}\,,\ \frac{\varepsilon}{1+3|g(x)|}\right\}.$$

Since g is continuous, we can choose $\delta_2 > 0$ such that

$$d(x,y) < \delta_2 \Longrightarrow |g(x) - g(y)| < \min\left\{\sqrt{\varepsilon/3}\,,\ \frac{\varepsilon}{1+3|f(x)|}\right\}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then

$$d(x,y) < \delta \Longrightarrow |f(x)g(x) - f(y)g(y)| =$$
$$= \big|\big(f(y)-f(x)\big)\big(g(y)-g(x)\big) + g(x)\big(f(y)-f(x)\big) + f(x)\big(g(y)-g(x)\big)\big|$$
$$\leq |f(y)-f(x)||g(y)-g(x)| + |g(x)||f(y)-f(x)| + |f(x)||g(y)-g(x)|$$
$$< \left(\sqrt{\varepsilon/3}\right)^2 + |g(x)|\frac{\varepsilon}{1+3|g(x)|} + |f(x)|\frac{\varepsilon}{1+3|f(x)|}$$
$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

(d) Exercise 9.4.1. □

We have given the above theorem its name because, for instance, if one thinks of the operations of addition and multiplication as functions from $\mathbb{R}^2$ to $\mathbb{R}$, then these functions are continuous, and this implies (b) and (c) of the theorem. (See Exercise 3.)

Corollary 9.7 *If $a_0, \ldots, a_n$ are fixed real numbers, then the function $p : \mathbb{R} \to \mathbb{R}$ defined by*

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

is continuous.

Proof. This follows from Example 8.6 and Theorem 9.6 by induction. □

In the next section, we will see some more techniques for proving that functions are continuous. In general, proofs of continuity can be tedious, and we will sometimes omit them in later chapters. However, the techniques of this chapter should enable you to fill in the details whenever you have doubts.

Exercises

1. Suppose f and g are continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$. Show that $f - g$ is continuous, where
$$(f - g)(x) \stackrel{\text{def}}{=} f(x) - g(x).$$

2. Fill in the details in the proof of Corollary 9.7.

3. In this exercise we will suggest a slightly different approach to proving the Continuous Operations Theorem.

   (a) Prove that the function $+ : \mathbb{R}^2 \to \mathbb{R}$ that takes (x_1, x_2) to $x_1 + x_2$ is continuous. *Hint:* let $\delta = \varepsilon/2$.

   (b) Suppose $f, g : \mathbb{R}^n \to \mathbb{R}$ are continuous. Prove that the function $f \times g : \mathbb{R}^n \to \mathbb{R}^2$ defined by
$$(f \times g)(x) \stackrel{\text{def}}{=} (f(x), g(x))$$
   is continuous. *Hint:* If
$$|f(x) - f(y)|, |g(x) - g(y)| < \frac{\varepsilon}{\sqrt{2}}$$
   then $d((f(x), g(x)), (f(y), g(y))) < \varepsilon$.

   (c) Use the Composition Theorem and parts (a) and (b) of this exercise to give an alternate proof of part (b) of the Continuous Operations Theorem.

4. Suppose that, for each $n \in \mathbb{Z}^+$ we are given a function $f_n : X \to Y$. We say that such a sequence of functions converges pointwise to $f : X \to Y$ if for every fixed $x \in X$ we have $\lim_{n\to\infty} f_n(x) = f(x)$. Suppose $X = Y = \mathbb{R}$ and each f_n of a sequence converging pointwise to f is continuous. Must f be continuous? Give a proof or counterexample.

5. What speculations do you have about the continuity of "polynomials of infinite degree" (otherwise known as power series)?