# Math 101: Solution Set #9

§$N8.1$ #6 **(a)** If $A$ is empty, then the problem is trivial. So assume $A$ is non-empty and pick a point $x \in A$. Then we know that $f(x) \in f(A)$ by definition of $f(A)$. However, we also know that $f^{-1}f(A) = \{x \in X \mid f(x) \in f(A)\}$ by definition of inverses. Therefore it is clear that $x \in f^{-1}f(A)$. Hence $x \in A \;\Rightarrow\; x \in f^{-1}f(A)$, and therefore $A \subset f^{-1}f(A)$.

Counter-example: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 1 \; \forall x \in \mathbb{R}$. Also let $A = \{4\}$. Then $f(A) = \{1\}$ and $f^{-1}f(A) = \mathbb{R}$. And since $\mathbb{R} \not\subseteq \{4\}$ then $f^{-1}f(A) \not\subseteq A$.

**(b)** Pick any $z \in ff^{-1}(A)$. We know that $ff^{-1}(A) = \{y \in Y \mid y = f(x) \text{ for some } x \in X \text{ with the property } x \in f^{-1}(A)\}$. I will abreviate this $\{f(x) \mid x \in f^{-1}(A)\}$. Therefore $z \in \{f(x) \mid x \in f^{-1}(A)\}$.
Hence there exists some $x_o \in f^{-1}A$ such that $z = f(x_o)$. Given this we know that $x_o \in \{x \in X \mid f(x) \in A\}$. And since we defined $f(x_o) = z$ then we know that $z \in A$.
Hence $z \in ff^{-1}(A) \;\Rightarrow\; z \in A$. Therefore $ff^{-1}(A) \subset A$.

Counter-example: Let $f$ be the same as in the previous counter-example. Let $A = \{1, 2\}$. Then $f^{-1}A = \mathbb{R}$ and then $ff^{-1}A = \{1\}$. Clearly $\{1, 2\} \not\subseteq \{1\}$. Hence $A \not\subseteq ff^{-1}A$.

§$N8.3$ #2 **(a)** Will we first show that $(gf)^{-1}A \subset f^{-1}(g^{-1}A)$.
To do this we first pick a point $c \in (gf)^{-1}A$. By definition $(gf)^{-1}A = \{x \in X \mid (gf)(x) \in A\}$. Since $c$ is in this set we know that $(gf)(c) = g(f(c)) \in A$.

We now recall that by definition $g^{-1}A = \{y \in Y \mid g(y) \in A\}$. From the above we know that $g(f(c)) \in A$ and therefore $f(c) \in g^{-1}A$.
Also by definition $f^{-1}(g^{-1}A) = \{x \in X \mid f(x) \in g^{-1}A\}$. But it then follows that $c \in f^{-1}(g^{-1}A)$.

Hence $c \in (gf)^{-1}A \;\Rightarrow c \in f^{-1}(g^{-1}A)$. Hence $(gf)^{-1}A \subset f^{-1}(g^{-1}A)$.

We now show that $f^{-1}(g^{-1}A) \subset (gf)^{-1}A$.
We pick any point $c \in f^{-1}(g^{-1}A) = \{x \in X \mid f(x) \in g^{-1}(A)\}$. Thus we know that $f(c) \in g^{-1}A$.
We also know that $g^{-1}A = \{y \in Y \mid g(y) \in A\}$. We already know that $f(c)$ is in this set, and hence we have that $g(f(c)) = (gf)(c) \in A$.
By definition, $(gf)^{-1}A = \{x \in X \mid gf(x) \in A\}$. Hence it is clear that $c \in (gf)^{-1}A$.

Therefore $c \in f^{-1}(g^{-1}A) \;\Rightarrow\; c \in (gf)^{-1}A$. Hence $f^{-1}(g^{-1}A) \subset (gf)^{-1}A$.

We have therefore shown that $f^{-1}(g^{-1}A) = (gf)^{-1}A$.

**(b)** The Inverse Continuity Theorem says that if $f : X \to Y$ is continuous then for evey closed set $B \subset Y$, $f^{-1}(B)$ is also closed.
Now suppose that we have two continuous functions $f : X \to Y$ and $g : Y \to Z$. Pick any closed set $C \subset Z$. Then $g^{-1}(C) \subset Y$ is closed, and then $f^{-1}(g^{-1}(C)) \subset X$ is also closed. But by part (a) we know that $f^{-1}(g^{-1}(C)) = (gf)^{-1}(C)$. Hence for all closed sets $C \subset Z$ we know that $(gf)^{-1}(C)$ is closed in $X$. Hence $gf : X \to Z$ is continuous.

#3 We know, from the notes, that our definition is equivalent to the Inverse Continuity Theorem. Hence I will prove that the "standard definition" is equivalent to ours. Throughout this problem I will assume that our function if $f : X \to Y$.

First: Std. Def. $\Rightarrow$ ICT

We must show that for any closed set $C \subset Y$, $f^{-1}C$ is closed in $X$.

Pick any closed set $C \in Y$. Now let $B = Y - C$. By definition $B$ is open. Hence by the "standard definition" $f^{-1}B$ is open. But $f^{-1}B = f^{-1}(Y - C) = f^{-1}Y - f^{-1}C = X - f^{-1}C$. And since this is open, $f^{-1}C$ is closed.

Second: ICT $\Rightarrow$ Std. Def.

We must show that for any open set $B \subset Y$, $f^{-1}B$ is open.

This proof works exactly the same way as the previous part. If you have questions *please* ask me!

#5 If we are considering the identity function $f : (X, K_1) \to (X, K_2)$ then by the definition of continuity, in order for $f$ to be continuous we must have that $fK_1A \subset K_2fA$ for all subsets $A \subset X$. But for ANY set $S \subseteq X$, $f(S) = S$. Hence the above condition on continuity is equivalent to requiring that $K_1A \subset K_2A$ for all $A \subset X$.

We could use this to order topologies by saying that $(X, K_1) < (X, K_2)$ iff the identity map $f : (X, K_1) \to (X, K_2)$ is continuous.