MATH 101 SOLUTION SET 8

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1. 6.1

1. *Prove that if $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are memebers of an ordered Archimedean field containing* $\mathbf{Z}^+$ *such that $a_1 \le a_2 \le \cdots \le \_2 \le b_1$, and if $b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n)$ for each n, then there exists only one point in $\cap_{n=1}^{\infty}[a_n, b_n]$.*

Assume a, b are members of the intersection with $a < b$. By induction it is easily proved that $b_n - a_n = \frac{1}{2^{n-1}}(b_1 - a_1)$. By the Archimedean property, as n ranges through $\mathbf{Z}^+$ this quantity grows arbitrarily small; in particular there will be an n such that $b_n - a_n < b - a$. But $[a, b] \subset [a_n, b_n]$ and we have a contradiction. Thus there can only be one point in the intersection.

2. *Is there a "principle of nested open intervals?"*

No. The open intervals $(0, 1/n)$ for $n = 1, 2, \ldots$ are open and nested, but their intersection is empty.

3. *Can two different elements of an ordered field F both be the least upper bounds of the same set $A \subset F$?*

No. If a and b are two least upper bounds of A, then since a is the l.u.b. and b is a u.b., we have by definition $a \le b$. But by symmetry $b \le a$, so $a = b$.

5. *Justify the assertion in the theorem that*

$$x = \sup\left\{\sum_{k=0}^{n} \frac{d_k}{10^k} \mid n \in \mathbf{Z}^+\right\}.$$

The supremum exists because each partial sum $\sum_{k=0}^{n} \frac{d_k}{10^k}$ is bounded by

$$d_0 + \sum_{k=1}^{n} \frac{9}{10^k} = d_0 + (1 - \frac{1}{10^n}) < d_0 + 1,$$

and since this bound is independent of n, the whole set is bounded. Let S be the supremum. x is defined to be the single element of the intersection

$$\cap_{n=0}^{\infty}\left[\sum_{k=0}^{n} \frac{d_k}{10^k}, \sum_{k=0}^{n} \frac{d_k}{10^k} + \frac{1}{10^n}\right],$$

so naturally it is at least as big as each partial sum $\sum_{k=0}^{n} \frac{d_k}{10^k}$; hence it is at least as big as the least upper bound: $x \ge S$. On the other hand, we have

$$S \ge \sum_{d=0}^{n} \frac{d_k}{10^k}$$

, so that

$$0 \leq x - S \leq x - \sum_{k=0}^{n} \frac{d_k}{10^k} \leq \frac{1}{10^n},$$

and since this is true for all n we have that $x - S$ must be zero.

2. 6.2

1. *Explain why the repeating decimal* $0.\bar{9}$ *must equal* 1.

Let $x = 0.\bar{9}$. By definition this is the least upper bound of the numbers $\sum_{k=1}^{n} \frac{9}{10^k} = (1 - \frac{1}{10^n})$. Since 1 is clearly an upper bound for these numbers, $x \leq 1$. On the other hand, if $y < 1$ is an upper bound for these numbers, let n be such that $\frac{1}{10^n} < 1 - y$; then y will be smaller than $1 - \frac{1}{10^n}$, contradiction. It follows that $x = 1$.

We can define the real numbers using decimals, but we have to take into account the ambiguity in representing a number of the form $n/10^k$ (where $n \in \mathbf{Z}$); the decimal expansion can either terminate in the digit d where $d < 9$ or it can repeat as $(d-1)999\dots$. The reals would have to be defined using equivalence classes in which these two representations were considered equivalent.

3. *State and prove a theorem that describes when the union of Dedekind cuts is a Dedekind cut.*

Many of you only considered finite unions, which is fine, but there's a theorem you can prove for arbitrary unions. Let $A_i \subset \mathbf{Q}$ be a collection of Dedekind cuts which is bounded from above; that is there exists a rational $a \in \mathbf{Q}$ such that $a \geq x$ for all x in each A_i. Then we claim the union $A = \cup_i A_i$ is also a Dedekind cut. There are three axioms we need to prove. Firstly, A is neither $\emptyset$ nor $\mathbf{Q}$, since the A_i are nonempty and since A is bounded from above. Second, if $a \in A$ and $s \in \mathbf{Q}$ has $s < r$, we need $s \in A$ as well. But this is true because if $a \in A$ we have $a \in A_i$ for some i; if $s < r$ then $s \in A_i$ hence $s \in A$ as required. Lastly, we need that A does not contain a largest element, but if it did contain a largest element $a \in A$, write $a \in A_i$; we would have that a was the largest element in A_i, which is a contradiction. Thus A is itself a Dedekind cut.

4. *Given* $A, B \subset \mathbf{R}$ *with* $x = \sup(A)$ *and* $y = \sup(B)$*, does it follow that* $x \pm y = \sup(A \pm B)$*?*

No. If $A = (0,1)$ and $B = (-1,1)$ then $x = y = 1$ but $A - B = (-1,2)$ whose supremum is 2. The result is true for addition, however. Furthermore if $x = \sup A$ and $y = \inf B$ then indeed $x - y = \sup(A - B)$.

6. *Show that if* $A \subset \mathbf{R}$ *has a least upper bound* $x = \sup(A)$*, then* x *belongs to* $\mathbf{K}(A)$.

Let $\varepsilon > 0$. We need to show that there's an $a \in A$ such that $x - a = |x - a| < \varepsilon$. If not, we have that $x - \varepsilon \geq a$ for all $a \in A$, so that $x - \varepsilon$ is also an upper bound for A. But this contradicts the minimality of x as a lower bound for A. It follows that $x \in \mathbf{K}(A)$.

7. *Show that the interval* $(0,1)$ *is uncountable.*

All of you used Cantor's diagonal argument, which is fine. Some care has to be taken, however, because of the ambiguity of the representation of a number as a decimal expansion mentioned in 6.2.1. Here's a different proof. Suppose the elements of $(0,1)$ could be listed $x_1, x_2, \dots$. We'll define a nested chain of intervals $[a_n, b_n]$ as follows. Let $[a_0, b_0] = [0,1]$. If $[a_0, b_0], \dots, [a_n, b_n]$ are already defined, let $[a_{n+1}, b_{n+1}]$ be any interval contained in $[a_n, b_n]$ which does not contain x_{n+1}. (It's not hard to see this can be done.) Then by PNCI the intersection $\cap_{n=0}^{\infty}[a_n, b_n]$ is nonempty, but none of its elements can be on the list $x_1, x_2, \ldots$ by construction. This is a contradiction and $(0, 1)$ must be uncountable.

3. 6.3

4. *A subset A of a topological space is called totally disconnected if its only connected components are singletons. Show that a discrete topological space is totally disconnected.*

If $A \subset X$ is a connected subset of a discrete space, assume it contains an element $a \in A$. Then we can write $A = \{a\} \cup (A - \{a\})$ as a union of two closed sets. (Discrete means that every subset is closed.) Since A is connected, this forces one of the subsets to be empty, namely $A - \{a\}$. Thus $A = \{a\}$ is a singleton.

Show that the Cantor Middle Thirds set is totally disconnected.

The only connected subsets of the reals are the intervals. We can show that there are no non-singleton intervals in C containing more than one element by showing that between any two elements of C there is a real number not in C. If $x, y \in C$, with $x < y$ the base three expansions of x and y may agree for the first few digits but eventually will disagree: $x = .d_0 d_1 \ldots d_n 0 \ldots$ and $y = .d_0 d_1 \ldots d_n 2 \ldots$. Then the point $.d_0 d_1 \ldots d_n 1$ will not be in C but will be between x and y.