Math 101: Solution Set #7

A.

§ $N4.2$ #1 Let $W \subset X$ be the set $\{x\}$. Assume that W is disconnected. Therefore there exists a separation (A, B) of W. However, since W only contains one point then either $a \in A$ and $a \in B$, or WLOG $a \in A$ and $B = \emptyset$. However, in the first case we would have that $A \cap B \neq \emptyset$ hence they could not form a separation. In the second case we have that $B = \emptyset$ so we cannot have a separation. This means that W must be connected.

#3 If (A, B) is a separation of Y then we know that $\mathbf{K}A \cap B = A \cap \mathbf{K}B = \emptyset$. We are also assuming that $A \cap W \neq \emptyset$ and $B \cap W \neq \emptyset$.

To prove that $(A \cap W), (B \cap W)$ is a separation of W we must show three things:
1) $(A \cap W) \cup (B \cap W) = W$
2) $\mathbf{K}(A \cap W) \cap (B \cap W) = \emptyset$
3) $(A \cap W) \cap \mathbf{K}(B \cap W) = \emptyset$

To show 1) we recall that $(A \cap W) \cup (B \cap W) = W \cap (A \cup B) = W \cap Y$. But we know that $W \subset Y$, hence $(A \cap W) \cup (B \cap W) = W \cap Y = W$.
To show 2) we recall that $\mathbf{K}(A \cap W) \subset \mathbf{K}A \cap \mathbf{K}W$. Hence we have that $\mathbf{K}(A \cap W) \cap (B \cap W) \subset \mathbf{K}A \cap \mathbf{K}W \cap W \cap B$. But we know that $\mathbf{K}W \cap W = W$, and hence $\mathbf{K}A \cap \mathbf{K}W \cap W \cap B = (\mathbf{K}A \cap B) \cap W = \emptyset \cap W = \emptyset$. Therefore $\mathbf{K}(A \cap W) \cap (B \cap W) \subset \emptyset \Rightarrow \mathbf{K}(A \cap W) \cap (B \cap W) = \emptyset$. The same proof goes through for 3) as did 2).

Hence $(A \cap W, B \cap W)$ is a separation of W.

#4 We are given that $W \subset X$ is connected. Assume however that $\mathbf{K}W$ is disconnected and that the separation is (A, B). I now claim that $A \cap W$ and $B \cap W$ are both non-empty. To show this assume that one were empty. So WLOG $A \cap W = \emptyset$. Since (A, B) is a separation of $\mathbf{K}W$, then we must have that $A \subset \mathbf{K}W - W$, and then it must be that $W \subset B$. But we then know that $\mathbf{K}W \subset \mathbf{K}B$. However, this implies that $A \cap \mathbf{K}B \neq \emptyset$ and then (A, B) would not be a valid separation. Hence it must be the case that $A \cap W$ and $B \cap W$ are both non-empty.

Given this, we know (by the last excercise) that $(A \cap W, B \cap W)$ form a separation of W. This is a contradiction since W is connected. Hence $\mathbf{K}W$ must also be connected.

#5 By the last excercies we know that since W is connected, $\mathbf{K}W$ must also be conncted. Now assume that there is a W' with $W \subset W' \subset \mathbf{K}W$ that is disconnected. Let the separation be (A, B). I claim that $A \cap W$ and $B \cap W$ are both nonempty. (I will leave the verification up to you b/c the proof is almost identical to the last excercise, and simply uses that fact that $W' - W \subset \mathbf{K}W$.) Given this we then have that $W \subset W'$ and $A \cap W$ and $B \cap W$ are both nonempty, therefore by a previous excercise we have a separation of W. This is a contradiction, hence W' must be connected.

B.

§ $N4.3$ #1 All of my examples will be in $\mathbb{R}$.
(a) A counterexample is $A = (0, 10)$ $B = (1, 2) \cup (5, 6)$.
(b) A counterexample is $A = (0, 10)$ $B = (5, 6)$.
(c) A counterexample is $A = (0, 1) \cup (1, 2)$ $B = \{1, 2\}$.

#2 Case $n = 1$: $\bigcup_{i=1}^{1} A_i = A_1$ which is connected by assumption.
Case $n = 2$: $\bigcup_{i=1}^{2} A_i = A_1 \cup A_2$. By assumption both are connected and $A_1 \cap A_2 \neq \emptyset$, hence by the Connceted Union Theorem (CUT) we know that $A_1 \cup A_2$ is connceted.
Assume that the theorem holds for the case of $n = k$ for any integer, k.
Case $n = k+1$: $\bigcup_{i=1}^{k+1} A_i = \bigcup_{i=1}^{k} A_i \cup A_{k+1}$. $\bigcup_{i=1}^{k} A_i$ is connected by the induction hypothesis. Also $A_k \cap A_{k+1} \neq \emptyset$ is connected by assumption. Then since $A_k \subset \bigcup_{i=1}^{k} A_i$ we know that $\bigcup_{i=1}^{k} A_i \cap A_{k+1} \neq \emptyset$. Hence by the CUT we know that $\bigcup_{i=1}^{k} A_i \cup A_{k+1} = \bigcup_{i=1}^{k+1} A_i$ is connected.

There IS a difference between the excercise and the CUT. The difference is that the CUT requires that $\bigcap_i A_i \neq \emptyset$ whereas this theorem only requires that $A_i \cap A_{i+1} \neq \emptyset$. Make sure you understand why these two requirements are different.

C.

§ $N4.4$ #1 1) Let $A = \{x\}$, then by the first problem on this set A is connected. Hence $x \sim x$.
2) If $x \sim y$, then there is a set $A \subset X$ that is connected with $x, y \in A$. But $x, y \in A$ is the same thing as $y, x \in A$, so $y \sim x$.
3) If $x \sim y$ and $y \sim z$ then there exist connected sets $A, B \subset X$ with $x, y \in A$ and $y, z \in B$. Hence $y \in A \cap B$ hence $A \cap B \neq \emptyset$. Therefore by the connected union theorem $A \cup B$ is connected, and we know that $x, z \in A \cup B$. Hence $x \sim z$.

#2 $C(x)$ is connected by definition. But assume that $C(x)$ is not the largest connected set containing x. Then there exists a set $D \subset X$ such that $x \in D$, D is connected and $D \not\subseteq C(x)$. If $D \not\subseteq C(x)$ then there exists some $y \in D$ such that $y \notin C(x)$. But if $y \notin C(x)$ then $x \nsim y$. By the definition of the equivalence relation this means that D must be disconnected since $x, y \in D$. This is a contradiction. Hence $C(x)$ is the largest connected set containing x.

#4 We know that $C(x)$ is closed, hence by a previous problem $\mathbf{K}C(x)$ is closed and by axiom C1 we know that $C(x) \subset \mathbf{K}C(x)$. This specifically tells us that $x \in \mathbf{K}C(x)$. But since we just proved that $C(x)$ is the largest connected set containing x that must mean that $\mathbf{K}C(x) \subset C(x)$. This combined with axiom C1 tells us that $C(x) = \mathbf{K}C(x)$.

#5 Eveyone got this one.