Math 101: Solution Set #5

A.

<u>§$W6.1, \#4$</u> **(a)** $A \times B$ has 6 elements in it, and therefore $\wp(A \times B)$ has $2^6 = 64$ elements.
(b) $|\wp(A)| = 2^3 = 8$ and $|\wp(B)| = 2^2 = 4 \;\Rightarrow |\wp(A) \times \wp(B)| = 8 \cdot 4 = 32$
(c) The set $A \bigcup B = \{1, 2, 3, 5\}$. This is not an ordered pair and therefore it is not in $A \times B$. Hence $|(A \times B) - (A \bigcup B)| = |A \times B| = 6$
(d) $A \bigcup B$ is the set of integers 1,2,3,5. $\wp(A \bigcup B)$ is a set of sets. It doesn't really make sense to subtract a set from an integer. Hence $|(A \bigcup B) - \wp(A \bigcup B)| = |A \bigcup B| = 4$.
(e) $|B \times B \times B \times B| = 2 \times 2 \times 2 \times 2 = 16$

<u>§$W6.2, \#2$</u> **(a)** This is not an equivalence relation because it is not symmetric. $A \subseteq B \;\not\Rightarrow B \subseteq A$.
(d) This is not an equivalence relation beacuse it is not transitive. For an example of this take $A = \{1, 2\}$ $B = \{2, 3\}$ and $C = \{3, 4\}$.
(e) This is an equivalence relation.

<u>§$W6.3, \#16$</u> **(a)** This will be a proof by contradiction. Assume $x, y \in A$ are both least elements with $x \neq y$. By definition of least element, we know that that means that $x \leq y$ and that $y \leq x$. But since we also know that A is a paritally ordered set, this implies that $x = y$ which is a contradiction. Hence a set can only have one least element. (Note that this same proof holds for greatest element.)
(b) $\forall y \in A \; (x \leq y)$ is logically equivalent to $\nexists y \in A \; \neg(x \leq y) \iff \nexists y \in A \; (x > y)$. (Again, the same proof holds for maximality).
(c) To show that there can be more then one minimal element we will consider the example where our set $A \subset \wp(\mathbb{N})$ and our ordering relation is that $E < F \iff E \subseteq F$. Let $A = \{\{1\}, \{2\}, \{1, 2\}\}$. Then $\{1\}$ is a miminal element and $\{2\}$ is a minimal element, but $\{1\} \neq \{2\}$. (Again, you can make a similar construction to show that there can be more then one maximal element.)

B.

<u>§$W8.3, \#12$</u> **(a)** If $a \equiv a' \pmod{n}$ and $b \equiv b' \pmod{n}$ then by definition $a = a' + xn$ and $b = b' + yn$ for some $x, y \in \mathbb{Z}$. Then we have that $a + b = a' + xn + b' + yn = a' + b' + (x + y)n \equiv a' + b' \pmod{n}$. Also we have that $ab = (a' + xn)(b' + yn) = a'b' + a'yn + b'xn + xyn^2 = a'b' + n(a'y + b'x + xyn) \equiv a'b' \pmod{n}$
(b) We can show that this is not well defined by giving a counterexample. Consider that we are working in the ring $\mathbb{Z}_4$. Then we know that $[-1] = [3]$. So by our definition of the absolute value function $|[3]| = [3]$ and $|[-1]| = [1]$, however we know that $[3] \neq [1]$. Therefore this function is not well defined.

<u>§$W8.3, \#14$</u> In $\mathbb{Z}_5$: $1^{-1} = 1,\; 2^{-1} = 3,\; 3^{-1} = 2,\; 4^{-1} = 4$. This is because $1 \cdot 1 = 1,\; 2 \cdot 3 = 6 \equiv 1,\; 3 \cdot 2 = 6 \equiv 1,$ and $4 \cdot 4 = 16 \equiv 1$

C.

<u>§$W9.5, \#1$</u> **(a)** We first let $\mathbb{Z}^* = \{(m,n)|m,n \in \mathbb{N}\}$. We now define an equivalence relation on this set. $(a,b) \sim (c,d) \iff a+d = b+c$. We can see that this is an equivalence on $\mathbb{Z}^*$ beacuse it is:
• Reflexive: $a+b = a+b$ and hence $(a,b) \sim (a,b)$
• Symmetric: If $a+b = c+d$ then $c+d = a+b$, hence $(a,b) \sim (c,d) \Rightarrow (c,d) \sim (a,b)$
• Transitive: Assume $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$. Then $a+d = b+c$ and $c+f = d+e$. Hence $a-b = c-d$ and $c-d = e-f$ therefore $a-b = e-f$ and hence $a+f = b+e \Rightarrow (a,b) \sim (e,f)$

We can now define $\mathbb{Z}$ as the set of all equivalence classes of $\sim$ on $\mathbb{Z}^*$

(b) Define $+$ as $(a_1,a_2)+(b_1,b_2) = (c_1,c_2) \iff c_1 = a_1+b_1$ and $c_2 = a_2+b_2$.
Define $\times$ as $(a_1,a_2)\times(b_1,b_2) = (c_1,c_2) \iff c_1 = a_1b_1+a_2b_2$ and $c_2 = a_1b_2+a_2b_1$.
Define $<$ as $(a_1,a_2) < (b_1,b_2) \iff a_1+b_2 < b_1+a_2$.

(c) For this part we choose x,y such that $(j',k') = (j,k) = x$ and $(m',n') = (m,n) = y$. By the definition of equivalent we therefore have that $j+k' = j'+k$ and that $m+n' = m'+n$.
• $(+)$: We have to show that $[(j,k)+(m,n)] = [(j',k')+(m',n')]$. By the definition of $+$ this is equivalent to showing that $[(j+m,k+n)] = [(j'+m',k'+n')]$. By the definition of equivalent this is the same as showing that $j+m+k'+n' = k+n+j'+m'$. But we already know that $j+k' = j'+k$ and that $m+n' = m'+n$, adding these two equations together gives us our desired result.
• $\times$ and $<$ can be shown in a similar manner.