MATH 101 SOLUTION SET 4

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A. *In the special case where A is a finite set, formulate and prove a result that justifies the notation 2^A for the power set of A.*

We claim that when A is finite, say $\#A = n$, then the power set of A has 2^n elements. We prove this by induction on n. If $n = 0$, $A = \emptyset$ and indeed $\wp(A) = \{\emptyset\}$ has $1 = 2^0$ elements. Assume the result for n, and let A be a set with $n+1$ elements. Let $x \in A$ be arbitrary, and note that the subsets of A can be partitioned into two categories: those that contain x, and those that don't. But each of these categories can be put into correspondence with $\wp(A - \{x\})$, which has 2^n elements by the inductive hypothesis. Thus $\wp(A)$ has $2^n + 2^n = 2^{n+1}$ elements.

B. 11. $\{0, 1\}$

13. a) $\{x > 1 \mid x \notin \mathbf{Z}\}$
b) $(1, \infty)$
c) $\mathbf{R} - \mathbf{Z}$
d) $\mathbf{R}$

C. We will show that the principle of induction implies the well-ordering principle. Let $S \subset \mathbf{Z}^+$ be a subset of the positive integers that has no least element. We will show that $S = \emptyset$. In particular we will prove the propositions

$$P(n): \ S \text{ contains no element less than or equal to } n$$

for each $n \in \mathbf{Z}^+$. $P(1)$ just says that $1 \notin S$. This is true, for if 1 were in S it would *a priori* be the least element of S. Now assume $P(n)$, so that no integer less than $n+1$ is in S. If $n+1$ were in S, then surely it would be the least element, so we must have $n+1 \notin S$, which proves $P(n+1)$. By induction, $P(n)$ is true for all n, so in particular $n \notin S$ for all n, implying $S = \emptyset$. □

Remark. Some of you tried to prove this by using induction on the cardinality of S, but this only shows that S has a least element when it is finite.

In fact, the other direction holds, and the well-ordering principle implies the axiom of induction. Let's prove this. Suppose WOP holds, and that $P(n)$ is a proposition with $P(1)$ true and $P(n) \implies P(n+1)$. Define

$$S = \left\{ n \in \mathbf{Z}^+ \mid P(n) \text{ is not true} \right\}.$$

Assume S is nonempty; then by WOP there is a least element $n \in S$. As $P(1)$ is true, we must have $n > 1$. Then $n - 1 \in \mathbf{Z}^+$ is less than n, the least element of S; therefore $n - 1 \notin S$, meaning that $P(n-1)$ is true, but by the inductive hypothesis $P(n)$ must also be true, so $n \notin S$, contradiction. Thus our assumption that $S \neq \emptyset$ was false. □

D.

1. All of you did this just fine.
2. Write $C = \cap_n A_n$ as it is defined in the notes, with $A_0 = [0,1]$ and $A_n = \frac{1}{3}A_{n-1} \cup (\frac{2}{3} + \frac{1}{3}A_{n-1})$ for $n > 0$. We claim that for all n, $1/4$ and $3/4$ lie in A_n. We'll prove this by induction. Surely $1/4, 3/4 \in A_0$.

Now suppose $1/4, 3/4 \in A_{n-1}$. Then

$$1/4 = \frac{1}{3}(3/4) \in \frac{1}{3}A_{n-1} \subset A_n$$

and

$$3/4 = \frac{2}{3} + \frac{1}{3}(1/4) = \frac{2}{3} + \frac{1}{3}A_{n-1} \subset A_n.$$

Thus $1/4 \subset A_n$ for all n, hence is in the intersection C.

Note that if we had tried to show that $1/4 \in A_n$ (alone) by induction, we would have failed. Sometimes in induction proofs it is necessary to strengthen the proposition $P(n)$ to make the proof work.

3. C is the set of all $x \in [0,1]$ which can be written in base three notation using only zeroes and twos.
4. C can be put into correspondence with $\wp(\mathbf{Z}^+)$ as follows. If $x \in C$, we can write a base 3 expansion $x = .a_0a_1\dots$ with $a_i \in \{0,2\}$. Now associate to x the subset $\{n \in \mathbf{Z}^+ \mid a_n = 0\}$. C will then have the same size as $\wp(\mathbf{Z}^+)$, which is uncountable.