Math 101 - Solution Set #1

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§N1.3

#1 **(a)** $\mathbf{K}A = \{x \in \mathbb{Z} | (\exists a \in A) a < x\}$
This is not a closure operator because it does not meet axiom C1. To prove this take the counter example of $A = \{1, 2, 3\}$. Then $\mathbf{K}A = \{x \in \mathbb{Z} | (\exists a \in A) a < x\} = \{2, 3, 4, \ldots\}$ Then it is easy to see that $A \not\subseteq \mathbf{K}A$.

(b) $\mathbf{K}A = \{x \in \mathbb{Z} | (\exists a \in A) a \leq x\}$
This is a closure operator, to prove it we check all four axioms.

C1: It is clear that for all a, $a \leq a$. Therefore for all $a \in A$, $a \in \mathbf{K}A$ because of the previous statement. So by definition of subset, $A \subseteq \mathbf{K}A$.

C2: By definition,

$$\begin{aligned}
\mathbf{K}(A \cup B) &= \{x \in \mathbb{Z} \mid (\exists c \in (A \cup B))\ c \leq x\} \\
\mathbf{K}A &= \{x \in \mathbb{Z} \mid (\exists a \in A)\ a \leq x\} \\
\mathbf{K}B &= \{x \in \mathbb{Z} \mid (\exists b \in B)\ b \leq x\}
\end{aligned}$$

$$\begin{aligned}
\text{Therefore: } \mathbf{K}A \cup \mathbf{K}B &= \{x \in \mathbb{Z} \mid ((\exists a \in A)\ a \leq x)\ OR\ ((\exists b \in B)\ b \leq x)\} \\
&= \{x \in \mathbb{Z} \mid ((\exists c \in A\ OR\ \exists c \in B)\ c \leq x)\} \\
&= \{x \in \mathbb{Z} \mid (\exists c \in (A \cup B))\ c \leq x\} \\
&= \mathbf{K}(A \cup B)
\end{aligned}$$

C3: $\mathbf{KK}A = \{y \in \mathbb{Z} \mid (\exists x \in \mathbf{K}A)\ x \leq y\}$. Now, by definition of $\mathbf{K}A$, we know that $x \in \mathbf{K}A \Rightarrow (\exists a \in A)\ a \leq x)$. We therefore have that

$$\begin{aligned}
\mathbf{KK}A &= \{y \in \mathbb{Z} \mid (\exists x \in \mathbf{K}A)\ x \leq y\} \\
&= \{y \in \mathbb{Z} \mid (\exists x \in \mathbb{Z})(\exists a \in A)\ a \leq x\ AND\ x \leq y\} \\
&= \{y \in \mathbb{Z} \mid (\exists x \in \mathbb{Z})(\exists a \in A)\ a \leq x \leq y\} \\
&= \{y \in \mathbb{Z} \mid (\exists a \in A)\ a \leq y\} \\
&= \mathbf{K}A
\end{aligned}$$

C4: By definition, $\mathbf{K}\varnothing = \{x \in \mathbb{Z} | (\exists a \in \varnothing) a \leq x\}$. However, we know that $\nexists a \in \varnothing$ and therefore $\mathbf{K}\varnothing = \varnothing$.

(c) $\mathbf{K}A = \{x \in \mathbb{Z} \mid (\exists a \in A)\ (\exists k \in \mathbb{Z})\ x = ka\}$
This is a closure operator, to prove it we check all four axioms.

C1: Since k ranges over all of $\mathbb{Z}$ then we know that we can consider $k = 1$. When $k = 1$ then $x = a$. Therefore $\forall a \in A,\ a \in \mathbf{K}A$ since $1 \in \mathbb{Z}$. Hence, $A \subseteq \mathbf{K}A$.

C2: By definition,

$$\begin{aligned} \mathbf{K}(A \cup B) &= \{x \in \mathbb{Z} \mid (\exists c \in (A \cup B))\ (\exists k \in \mathbb{Z}\ c \leq x\} \\ \mathbf{K}A &= \{x \in \mathbb{Z} \mid (\exists a \in A)\ (\exists k \in \mathbb{Z})\ x = ka\} \\ \mathbf{K}B &= \{x \in \mathbb{Z} \mid (\exists b \in B)\ (\exists k \in \mathbb{Z})\ x = kb\} \end{aligned}$$

$$\begin{aligned} \text{Therefore: } \mathbf{K}A \cup \mathbf{K}B &= \{x \in \mathbb{Z} \mid ((\exists a \in A)(\exists k \in \mathbb{Z})\ x = ka)\ OR\ ((\exists b \in B)(\exists k \in \mathbb{Z})\ x = kb)\} \\ &= \{x \in \mathbb{Z} \mid (\exists c \in A\ OR\ \exists c \in B)(\exists k \in \mathbb{Z})\ x = kc\} \\ &= \{x \in \mathbb{Z} \mid (\exists c \in (A \cup B)(\exists k \in \mathbb{Z})\ x = kc\} \\ &= \mathbf{K}(A \cup B) \end{aligned}$$

C3: By Definition: $\mathbf{KK}A = \{y \in \mathbb{Z} \mid (\exists x \in \mathbf{K}A)(\exists l \in \mathbb{Z})\ y = lx\}$. Now we note that $x \in \mathbf{K}A \Rightarrow (\exists a \in A)(\exists k \in \mathbb{Z})\ x = ka$.

$$\begin{aligned} \text{Thus: } \mathbf{KK}A &= \{y \in \mathbb{Z} \mid (\exists a \in A)(\exists k \in \mathbb{Z})(\exists l \in \mathbb{Z})\ x = ka\ AND\ y = lx\} \\ &= \{y \in \mathbb{Z} \mid (\exists a \in A)(\exists k, l \in \mathbb{Z})\ y = lka\} \\ &= \{y \in \mathbb{Z} \mid (\exists a \in A)(\exists m \in \mathbb{Z})\ y = ma\} \\ &= \mathbf{K}A \end{aligned}$$

C4: $\mathbf{K}\varnothing = \{x \in \mathbb{Z} \mid (\exists a \in \varnothing)\ (\exists k \in \mathbb{Z})\ x = ka\}$.
But we know that $\nexists a \in \varnothing$, and therefore $\mathbf{K}\varnothing = \varnothing$.

(d) $\mathbf{K}A = \{x \in \mathbb{Z} \mid (\exists a_1 \in A)\ (\exists a_2 \in A)\ a_1 \leq x \leq a_2\}$
This is not a closure operator because it fails to satisfy the C2 axiom. To prove this we consider the following counterexample:
Let, $A = \{1, 2\}$ $B = \{5, 6\}$ Then $\mathbf{K}A = \{1, 2\}$ and $\mathbf{K}B = \{5, 6\}$.
Therefore $\mathbf{K}A \cup \mathbf{K}B = \{1, 2, 5, 6\}$.
However, $\mathbf{K}(A \cup B) = \mathbf{K}(\{1, 2, 5, 6\}) = \{1, 2, 3, 4, 5, 6\} \neq \mathbf{K}A \cup \mathbf{K}B$.

#2 **C1:** If A is finite, then $\mathbf{K}A = A$, and hence $A \subseteq \mathbf{K}A$.

If A is inifinite then $\mathbf{K}A = A \cup \{\infty\}$, and hence $A \subseteq \mathbf{K}A$ (by defn. of union).

C2: If A, B are both finite then $\mathbf{K}A = A$ and $\mathbf{K}B = B$. Hence $\mathbf{K}A \cup \mathbf{K}B = A \cup B$.
Also, if A and B are finite, then so is $A \cup B$. Hence $\mathbf{K}(A \cup B) = A \cup B$.

If A and B are both infinite, then so is $A \cup B$. Hence $\mathbf{K}A = A \cup \{\infty\}$, $\mathbf{K}B = B \cup \{\infty\}$, and $\mathbf{K}(A \cup B) = A \cup B \cup \{\infty\}$. So we have that $\mathbf{K}A \cup \mathbf{K}B = A \cup \{\infty\} \cup B \cup \{\infty\} = A \cup B \cup \{\infty\} = \mathbf{K}(A \cup B)$.

Now we consider the case where one of the sets is infinite and the other is finie. So, WLOG assume that A is infinite and B is finite. In that case $\mathbf{K}A = A \cup \{\infty\}$ and $\mathbf{K}B = B$. Also, if A is infinite, then $A \cup B$ is also infinite. Hence $\mathbf{K}(A \cup B) = A \cup B \cup \{\infty\}$. We also have that $\mathbf{K}A \cup \mathbf{K}B = A \cup \{\infty\} \cup B = A \cup B \cup \{\infty\} = \mathbf{K}(A \cup B)$

C3: If A is fininte then $\mathbf{K}A = A$ and hence $\mathbf{KK}A = \mathbf{K}A$.

If A is infinite then $\mathbf{K}A = A \cup \{\infty\}$. Then $\mathbf{KK}A = \mathbf{K}(A \cup \{\infty\})$. We also know that $A \cup \{\infty\}$ is infinite since A is. Hence $\mathbf{K}(A \cup \{\infty\}) = (A \cup \{\infty\}) \cup \{\infty\} = A \cup \{\infty\} = \mathbf{K}A$

C4: $\varnothing$ is a finite set, and therefore $\mathbf{K}\varnothing = \varnothing$.

Hence, $\mathbf{K}$ is a valid closure operator on $\mathbb{L}$.

§N1.4

#1 **(a)** $\mathbf{K}(A_1 \cup A_2 \cup A_3) = \mathbf{K}((A_1 \cup A_2) \cup A_3) = \mathbf{K}(A_1 \cup A_2) \cup \mathbf{K}(A_3)$ by axiom C2. Again by axiom C2, this is equal to $\mathbf{K}A_1 \cup \mathbf{K}A_2 \cup \mathbf{K}A_3$

(b) $\mathbf{K}(A_1 \cup A_2 \cup A_3 \cup A_4) = \mathbf{K}((A_1 \cup A_2 \cup A_3) \cup A_4) = \mathbf{K}(A_1 \cup A_2 \cup A_3) \cup \mathbf{K}A_4$ by axiom C2. From part **(a)** we know that $\mathbf{K}(A_1 \cup A_2 \cup A_3) = \mathbf{K}A_1 \cup \mathbf{K}A_2 \cup \mathbf{K}A_3$. And we therefore have that $\mathbf{K}(A_1 \cup A_2 \cup A_3 \cup A_4) = \mathbf{K}A_1 \cup \mathbf{K}A_2 \cup \mathbf{K}A_3 \cup \mathbf{K}A_4$.

#2 **(a)** This statement is not always true. For a counter example let us consider any nonempty space X under the trivial topolgoy. (So $\mathbf{K}A = A$ if $A \neq \varnothing$, and $\mathbf{K}\varnothing = \varnothing$. We showed in class that this is a valid closure operator.) Let us now pick two nonempty subsets $A, B \subset X$ such that $A \not\subset B$. Then since $A - B \neq \varnothing$, $\mathbf{K}(A - B) = X$. However, $\mathbf{K}A = X = \mathbf{K}B$. Hence $\mathbf{K}A - \mathbf{K}B = \varnothing$. Since X is nonempty, then $X \not\subset \varnothing$. Hence **(a)** is false.

(b) We prove the hint. First proving that $C - D \subset E \Rightarrow C \subset (D \cup E)$.
Pick any $x \in (C - D)$, then by defn. of subset, $x \in E$. I now claim that every $y \in C$ is either in D or in E. So we pick any $y \in C$. If $y \in D$ then we are done. If $y \notin D$ then by definition $y \in (C - D)$ and therefore $y \in E$. Hence every $y \in C$ is either in D or in E. Therfore by definition $C \subset (D \cup E)$.

We now prove that $C \subset (D \cup E) \Rightarrow C - D \subset E$.
By definition of subset and union, every $x \in C$ must either be in D or in E. Therefore if you have an x that is in C but is not in D it must necessarily be in E. Hence we have that $C - D \subset E$.

We now proceed to prove that $\mathbf{K}A - \mathbf{K}B \subset \mathbf{K}(A - B)$

$A - B \subset \mathbf{K}(A - B)$	Axiom C1
$A \subset \mathbf{K}(A - B) \cup B$	From the hint
$\mathbf{K}A \subset \mathbf{K}(\mathbf{K}(A - B) \cup B)$	Thm. 1.9 in the notes
$\mathbf{K}A \subset \mathbf{KK}(A - B) \cup \mathbf{K}B$	Axiom C2
$\mathbf{K}A \subset \mathbf{K}(A - B) \cup \mathbf{K}B$	Axiom C3
$\mathbf{K}A - \mathbf{K}B \subset \mathbf{K}(A - B)$	From the hint

§W2.1 (Note: #3, #5, and #6 can all be checked with truth tables, I will not go through all of it here, but if you have questions please ask me.)

#3 **(a)** Neither.
(b) Tautology.
(c) Tautology.
(d) Neither.
(e) Tautology.
(f) Neither.

#5 **(a)** iii
(b) ii
(c) v
(d) vi
(e) vii

#6 **(a)** $\rightarrow$
(b) $\wedge$
(c) $\vee$
(d) $\rightarrow$
(e) $\wedge$

§W2.2

#4 No one seemed to have any trouble with making up the sentences in here, so I will not go through all of the answers. One thing to note is that in part **(c)** the sentence is "It's necessary to give a baby nourishing food in order for it to grow up healthy." This means that P = The baby grows up healthy, while Q = You give the baby nourishing food.

#7 Some people did this problem by making up little sentences, and others wrote logical statements. Either was acceptable. For the sake of the solution set I will give the answers in logical statement form. This are just possible answers and are definately not the only correct answers.
(a) $P \rightarrow (Q \vee \sim Q)$ (Since $(Q \vee \sim Q)$ is always true, then this statement is always true, but the converse doesn't have to be since P can be false.)
(b) This is not possible since a conditional statment is equivalent to its contrapositive.
(c) $Q \wedge \sim Q \rightarrow P \vee \sim P$ (This will always be true and yet the inverse will always be false.)
(d) There were two different ways to view this problem. One method was to take the conventional definition of False, and the other was to use how the book defined it for this problem, that is that False means *not* necessarily true.
If you took False to mean strictly False, then this part was not possible. The reason is that if $P \rightarrow Q$ is false, then it must be the case that P is True and Q is False. But if that is so then $Q \rightarrow P$ must be true. Hence the request is not possible.
If you took False the way the book defined it then one possible answer might be $P \rightarrow \sim P$.

§W5.2

#6 **(a)** This statement is not true, and to show not we will give a counter example. Take A to be any nonempty set, and take $B = C = \varnothing$. Then $A \cup (B - C) = A \cup \varnothing = A$. However, $(A \cup B) - (A \cup C) = (A \cup \varnothing) - (A \cup \varnothing) = A - A = \varnothing$. By defnition $A \nsubseteq \varnothing$ and hence $A \cup (B - C) \nsubseteq (A \cup B) - (A \cup C)$.

We now prove that $A \cup (B - C) \supseteq (A \cup B) - (A \cup C)$.
Pick any $x \in (A \cup B) - (A \cup C)$, then $x \in (A \cup B) \wedge x \notin (A \cup C)$
$\Rightarrow (x \in A \vee x \in B) \wedge (x \notin A \wedge x \notin C)$
Since it is clear that x must either be in A or B then let us test what happens if $x \in A$. If this is the case then we get a contradiction because we are told that $x \notin A$. Therefore according to the first half of this statement, $x \in B$. The second half tells us that $x \notin C$. And $(x \in B) \wedge (x \notin C) \Rightarrow x \in (B - C)$ by definition of $-$. Now, by definition of union, it must be the case that $x \in (B - C) \cup A$.

(b) This statement is not tru, and we can give a counter example to show so. Take $A = \{1, 2, 3\}$ $B = \{2, 3\}$ $C = \{3\}$.
Then $A - (B \cup C) = \{1, 2, 3\} - \{2, 3\} = \{1\}$
However, $(A - B) \cup (A - C) = \{1\} \cup \{1, 2\} = \{1, 2\}$
So we can see that $(A - B) \cup (A - C) \nsubseteq A - (B \cup C)$.

We now prove that $(A - B) \cup (A - C) \supseteq A - (B \cup C)$.
Thm. 5.2 tells us that $A - (B \cup C) = (A - B) \cap (A - C)$. Therefore is we pick any $x \in A - (B \cup C)$ we know that $x \in (A - B)$. Therefore by definition of union, $x \in (A - B) \cup (A - C)$. (We can see this because by definition, $x \in A \Rightarrow x \in A \cup Y$ for any set Y.)

(c) This statement is true, so we prove it in both directions.
$(\Rightarrow)$: We are given that $A \subseteq B$. By definition this means that $x \in A \Rightarrow x \in B$. Not let us assume that $A - B \neq \varnothing$. That would mean $\exists x \in (A - B) \Rightarrow \exists x$ s.t. $(x \in A \wedge x \notin B) \Rightarrow \exists x$ s.t. $(x \in A \nRightarrow x \in B)$. This directly contradicts the definition of $A \subseteq B$ as stated above. Hence it must be the case that $A - B = \varnothing$.

$(\Leftarrow)$: We are given that $A - B = \varnothing$. This implies that $\nexists x \in (A - B) \Rightarrow \nexists x \in A$ s.t. $x \notin B \Rightarrow \forall x \in A,\ x \in B \Rightarrow A \subseteq B$.

(d) This statement is not true, and to prove so we give a counter example:
Let $A = \{2, 3\}$ $B = \{1, 2\}$, $C = \{3, 4\}$. Then $B \cup C = \{1, 2, 3, 4\}$ and clearly $A \subseteq B \cup C$ however $A \nsubseteq B$ and $A \nsubseteq C$.

#13 This theorem is false, and the proof goes wrong in the "reverse direction" when it ignores the case of $x = b$. If $x = b$ then $x \notin (a, b)$ and $x \notin (b, c)$ by definition of an open interval. Therefore $x \notin (a, b) \cup (b, c)$.

#14 To the best of my knowledge there are no problems with this theorem and proof.