Final, Mathematics 101

Solutions

Problem 1. Show that for any three nonempty sets X, Y, Z

$$\left|\left(X^Y\right)^Z\right| = \left|X^{(Y\times Z)}\right|.$$

Solution. Let us construct a bijection F between $\left|\left(X^Y\right)^Z\right|$ and $\left|X^{(Y\times Z)}\right|$.
Let $f \in \left|X^{(Y\times Z)}\right|$. Define $F(f) \in \left|\left(X^Y\right)^Z\right|$ by

$$F(f)(z)(y) = f(y, z).$$

To show that F is bijection we construct F^{-1}. For $g \in \left|\left(X^Y\right)^Z\right|$ we define $F^{-1}(g)(y, z) = g(z)(y)$.

Problem 2. Show that any set of pairwise nonintersecting disks on the plane is countable.

Solution. You can just select a point with both coordinates rational in every disc. This defines an injection from the set of disks to $\mathbb{Q} \times \mathbb{Q}$. So our set has the same cardinality as a subset of $\mathbb{Q} \times \mathbb{Q}$, and thus countable.

Problem 3. Construct a set of sets A such that $\cap A = \emptyset$, but for any countable subset $B \subset A$ we have $\cap B \neq \emptyset$.

Solution. Let for every $r \in \mathbb{R}$ $X_r = \mathbb{R} - \{r\}$. Let $A = \{X_r, r \in \mathbb{R}\}$. Note that $\cap A = \mathbb{R} - \mathbb{R} = \emptyset$, but for any countable $B = \{X_{r_1}, X_{r_2}, \dots\}$ we have $\cap B = \mathbb{R} - \{r_1, r_2, \dots\} \neq \emptyset$, since $\mathbb{R}$ is not countable.

Problem 4. Show that any two cycles σ_1 and σ_2 of the same length in S_n are conjugate, i.e. there exists $g \in S_n$ such that $\sigma_1 = g\sigma_2 g^{-1}$.

Solution. Let $\sigma_1 = (a_1\ a_2\ \dots a_k)$, $\sigma_2 = (b_1\ b_2\ \dots b_k)$. Define $g(a_i) = b_i,\ i = 1, 2, \dots, k$, and define it arbitrarily (but making it bijection!) for all other z. Than g conjugates σ_1 and σ_2.

Problem 5. Show that if N and M are normal subgroups of a group G with $M \cap N = \{e\}$ then every two elements $m \in M,\ n \in N$ commute.

Solution. Consider the element $x = mnm^{-1}n^{-1}$. On one hand, $mnm^{-1} \in N$ by normality of N, so $x = (mnm^{-1})n^{-1} \in N$. On the other hand, $nm^{-1}n^{-1} \in M$ by normality of M, so $x = m(nm^{-1}n^{-1}) \in M$. Thus $x \in M \cap N$, so $x = e$. So $mnm^{-1}n^{-1} = e$, and thus $mn = nm$.

Problem 6. Describe all cyclic subgroups of D_n.

Solution. D_n consists of n flips. Each of them generates a cyclic group of order 2. All other elements of D_n are rotations by $\frac{2\pi k}{n}$, $k = 0, 1, \dots, n-1$, which themself form a cyclic group of order n. Thus its subgroups (all cyclic!) are generated by divisors of n.

Problem 7. Prove that a circle is not homeomorphic to an annulus.

Solution. Suppose that f is a homeomorphism between a circle and an annulus. pick any two (distinct) points x, y on the circle. Then f is a hgomeomorphism between compliment of this two points and compliment of their images. But compliment of any two points on the circle is not connected, and compliment of any two points on the annulus is connected. So we get a contradiction, and such an f does not exist.