A curiosity in measure theory

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I ran across the following curiosity when teaching a graduate course in real analysis several years ago. I submitted it as a problem to the American Mathematical Monthly; the Problems editor rejected it on the advice of both referees — one of whom thought the problem unfairly difficult, the other thought it trivial! What do you think?

As usual in this subject, we assume the Axiom of Choice, and say that a subset of $\mathbb{R}$ is "negligible" if it is Lebesgue measurable with measure zero. A family of subsets of some set X is said to be "totally ordered" if for every A, B in the family either $A \subseteq B$ or $B \subseteq A$.

Q: Suppose $\mathcal{F}$ is a totally ordered family of negligible subsets of the unit interval $[0, 1]$. Must $\cup_{A \in \mathcal{F}} A$ be negligible?

A: No. Proof: Assume that in fact $\cup_{A \in \mathcal{F}} A$ is negligible for any such $\mathcal{F}$. Then the family of *all* negligible subsets of $[0, 1]$ would satisfy the hypotheses of Zorn's Lemma! There would then be a maximal negligible subset. But this is absurd. Indeed, suppose S is a maximal negligible subset of $[0, 1]$. If there were $x \in [0, 1]$ not contained in S, then $S \cup \{x\}$ would be a negligible subset of $[0, 1]$ strictly containing S. Hence there is no such x, and $S = [0, 1]$. But it is known that $[0, 1]$ is not negligible. $\diamondsuit$

Remark: This proof is curiously nonconstructive even by the standards of the subject. It is true that this could be remedied by working through the proof of Zorn's Lemma to "construct" an example of a totally ordered family $\mathcal{F}$ of negligible subsets of $[0, 1]$ whose union, call it U, is not negligible. But this still leaves the question of how "big" this U can be. For instance, can U be all of $[0, 1]$? If we also assume the Continuum Hypothesis, the answer is yes: we can then use a bijection of $[0, 1]$ with the smallest uncountable ordinal to impose a total order $\prec$ on $[0, 1]$ such that $I_x := \{y \in [0, 1] : y \preceq x\}$ is countable — and thus negligible — for all $x \in [0, 1]$, and set $\mathcal{F} = \{I_x : x \in [0, 1]\}$. Without the CH assumption, I do not know the answer.