

Why is π^2 so close to 10?
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The ancient and still useful approximation $\pi \doteq \sqrt{10}$ may appear to be a mere coincidence. But if we accept Euler's theorem that

$$\pi^2 = 6\zeta(2) = 6 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

then we may easily deduce that $\pi^2 < 10$, and estimate the error in the approximation. Indeed we have

$$\zeta(2) = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} < 1 + \sum_{n=2}^{\infty} \frac{4}{4n^2 - 1},$$

and the sum telescopes because

$$\frac{4}{4n^2 - 1} = \frac{4}{(2n-1)(2n+1)} = \frac{2}{2n-1} - \frac{2}{2n+1} = \frac{2}{2n-1} - \frac{2}{2(n+1)-1}.$$

We conclude

$$\zeta(2) < 1 + \left(\left(\frac{2}{3} - \frac{2}{5} \right) + \left(\frac{2}{5} - \frac{2}{7} \right) + \left(\frac{2}{7} - \frac{2}{9} \right) + \dots \right) = 1 + \frac{2}{3} = \frac{5}{3},$$

so $\pi^2 < 6 \cdot (5/3) = 10$ as claimed; and the error is reasonably small because

$$\frac{5}{3} - \zeta(2) = \sum_{n=2}^{\infty} \left(\frac{4}{4n^2 - 1} - \frac{1}{n^2} \right) = \sum_{n=2}^{\infty} \frac{1}{n^2(4n^2 - 1)},$$

a sum whose first term is $1/60$ and whose further terms are much smaller yet. An even better and still memorable approximation to the actual value $\pi^2 = 9.8696044\dots$ is obtained by extracting a second term from the sum for $\zeta(2)$ before using $1/n^2 < 4/(4n^2 - 1)$: we find

$$\pi^2 = 6\zeta(2) < 6 \left(1 + \frac{1}{4} + \frac{2}{5} \right) = 9.9.$$