

The power series for the inverse function of ye^{-y}
Noam D. Elkies

For small x let $y(x)$ be the small solution of $ye^{-y} = x$. Abel showed that this function has the curious power series

$$y = \sum_{n=1}^{\infty} n^{n-1} \frac{x^n}{n!}.$$

One standard application of this is the enumeration of trees on n labeled vertices (there are n^{n-2}). Abel's power series is usually proved using the residue theorem. The following derivation uses only elementary calculus, and as an added bonus gives the power series expansion of y^β for all β .

Differentiate both sides of $ye^{-y} = x$ to find

$$1 = (1 - y)e^{-y}y' = (1 - y)\frac{x}{y}y'.$$

Multiply by y^β to obtain

$$y^\beta = x(y^{\beta-1} - y^\beta)y' = x \frac{d}{dx} \left(\frac{y^\beta}{\beta} - \frac{y^{\beta+1}}{\beta+1} \right) \quad (1)$$

(for $\beta \neq 0, -1$). Now recall that the operator $x d/dx$ takes any power series $\sum_n a_n x^n$ to $\sum_n n a_n x^n$. Thus if for any c we expand y^c in a power series

$$y^c = \sum_{m=0}^{\infty} a_{m+c}(c) x^{m+c}$$

then our identity (1) amounts to

$$a_{m+\beta}(\beta) = (m + \beta) \left(\frac{a_{m+\beta}(\beta)}{\beta} - \frac{a_{m+\beta}(1 + \beta)}{1 + \beta} \right),$$

which simplifies to the recursion

$$a_{m+\beta}(\beta) = \frac{\beta}{1 + \beta} \frac{m + \beta}{m} a_{m+\beta}(1 + \beta).$$

Applying this m times, we find

$$a_{m+\beta}(\beta) = \frac{\beta}{m + \beta} \frac{(m + \beta)^m}{m!} a_{m+\beta}(m + \beta).$$

But $a_{m+\beta}(m + \beta)$ is the leading coefficient of $y^{m+\beta}$, and thus equals 1. We conclude that

$$a_{m+\beta}(\beta) = \frac{\beta(m + \beta)^{m-1}}{m!}.$$

In particular, taking $\beta = 1$ and $m = n - 1$ we find that the x^n coefficient of y is $n^{n-1}/n!$, as claimed.