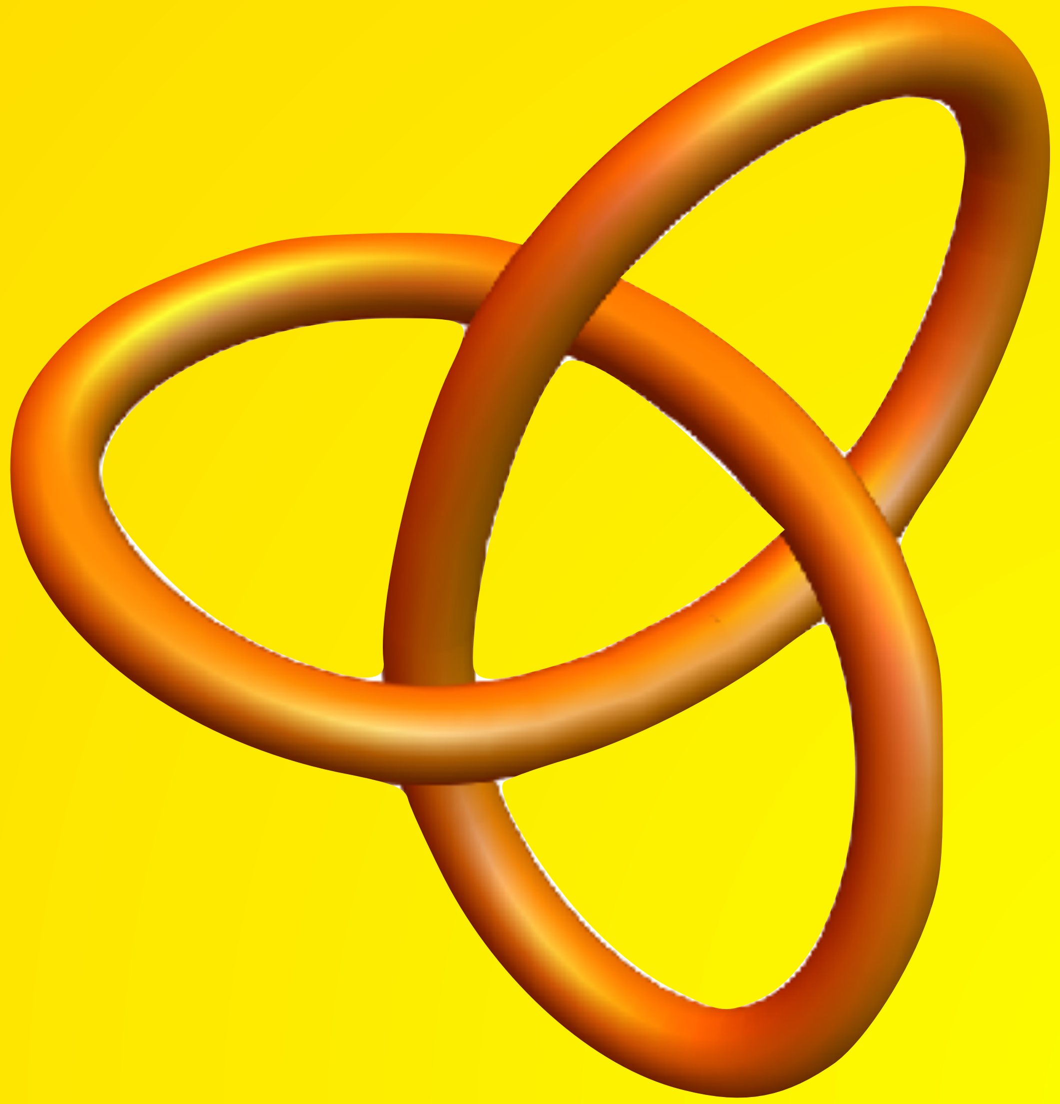
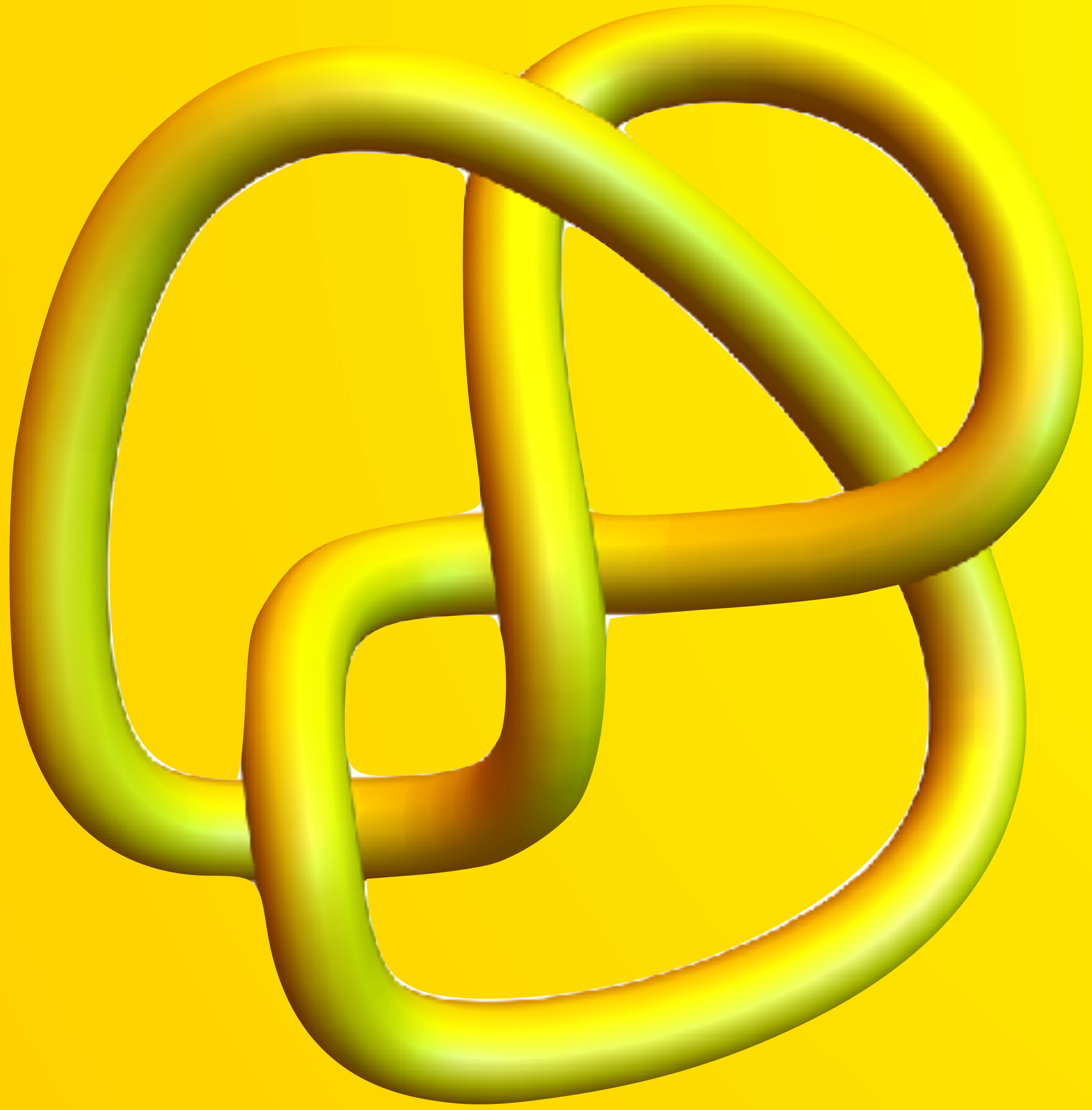


Lecture 15



Curves

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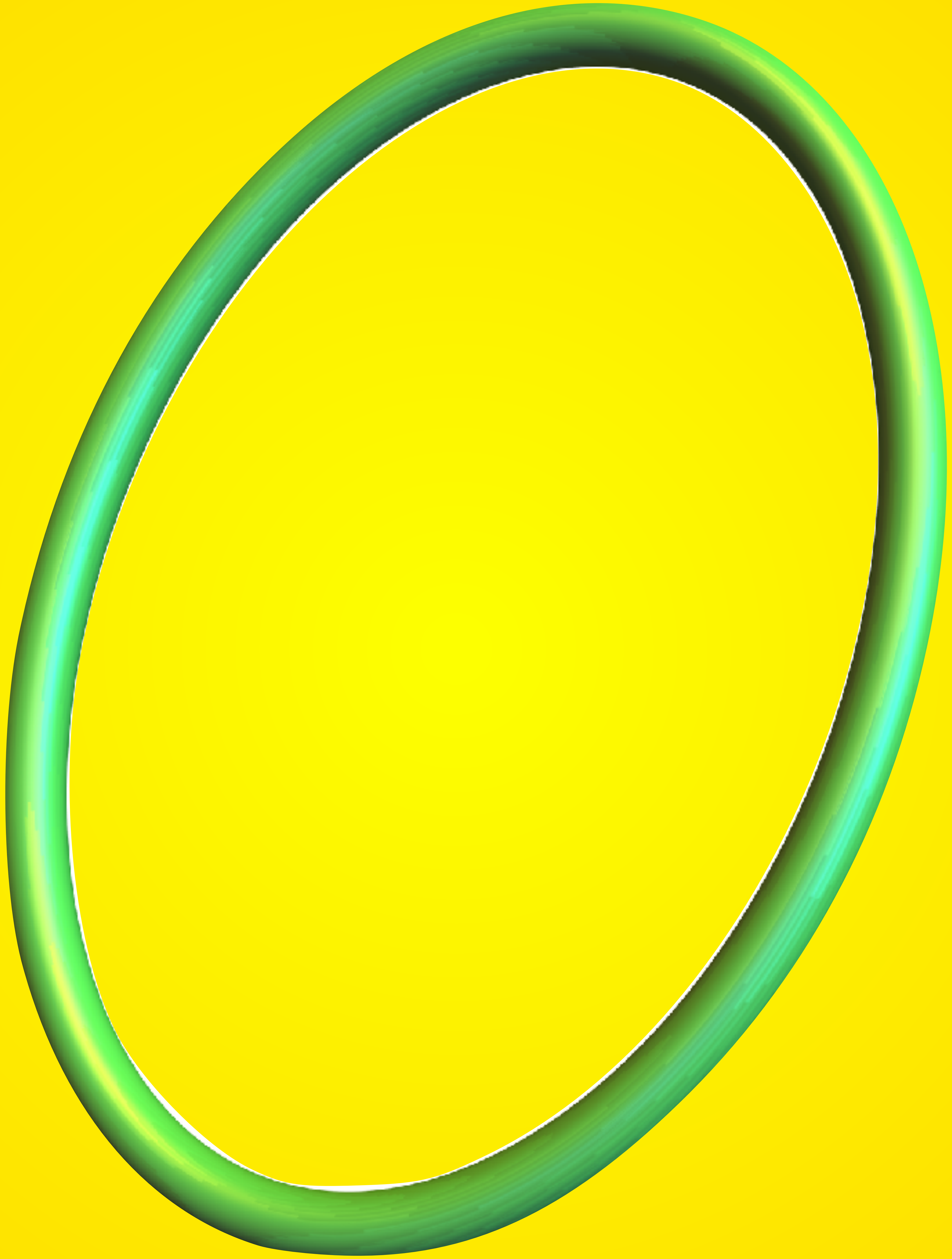
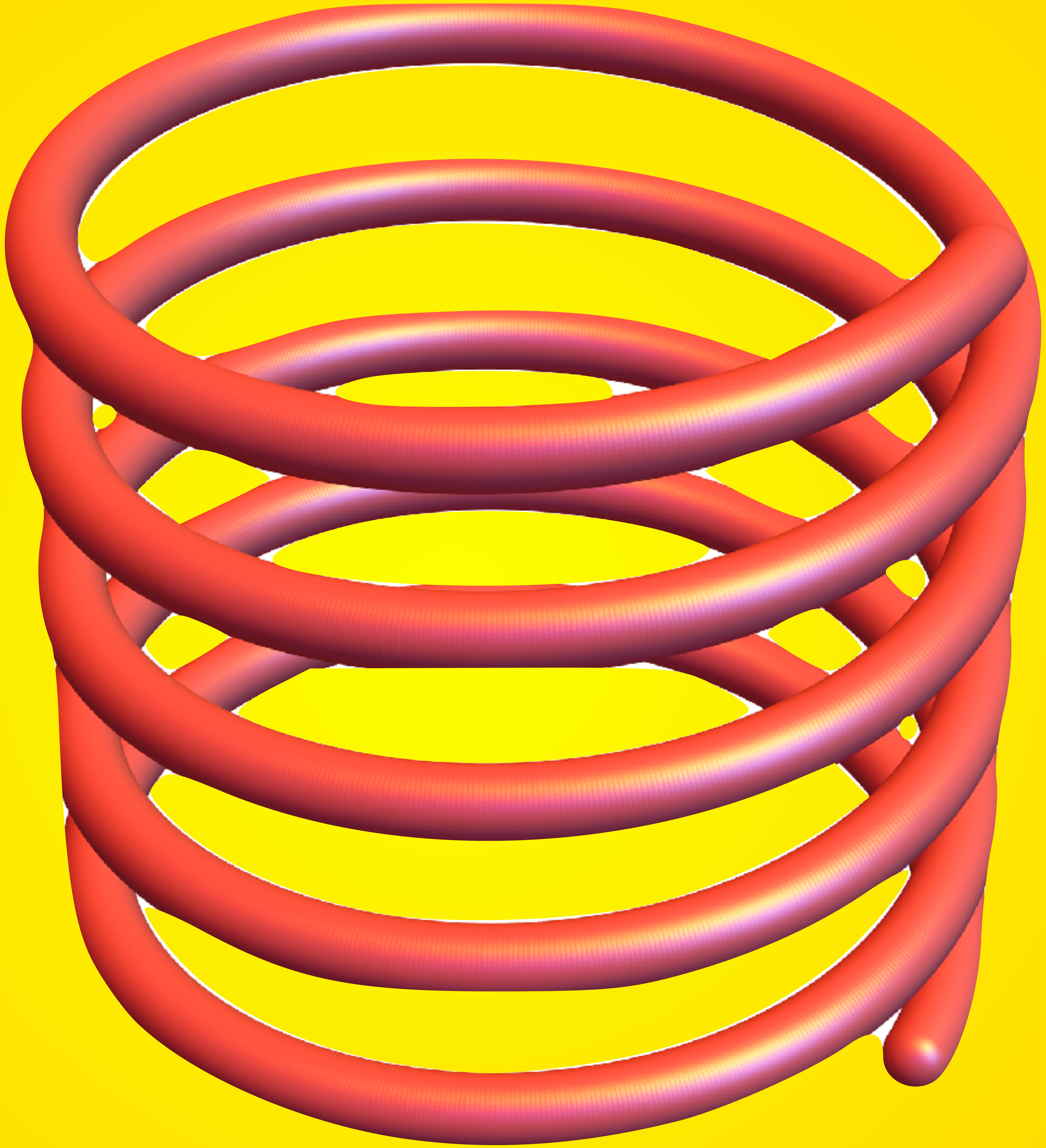
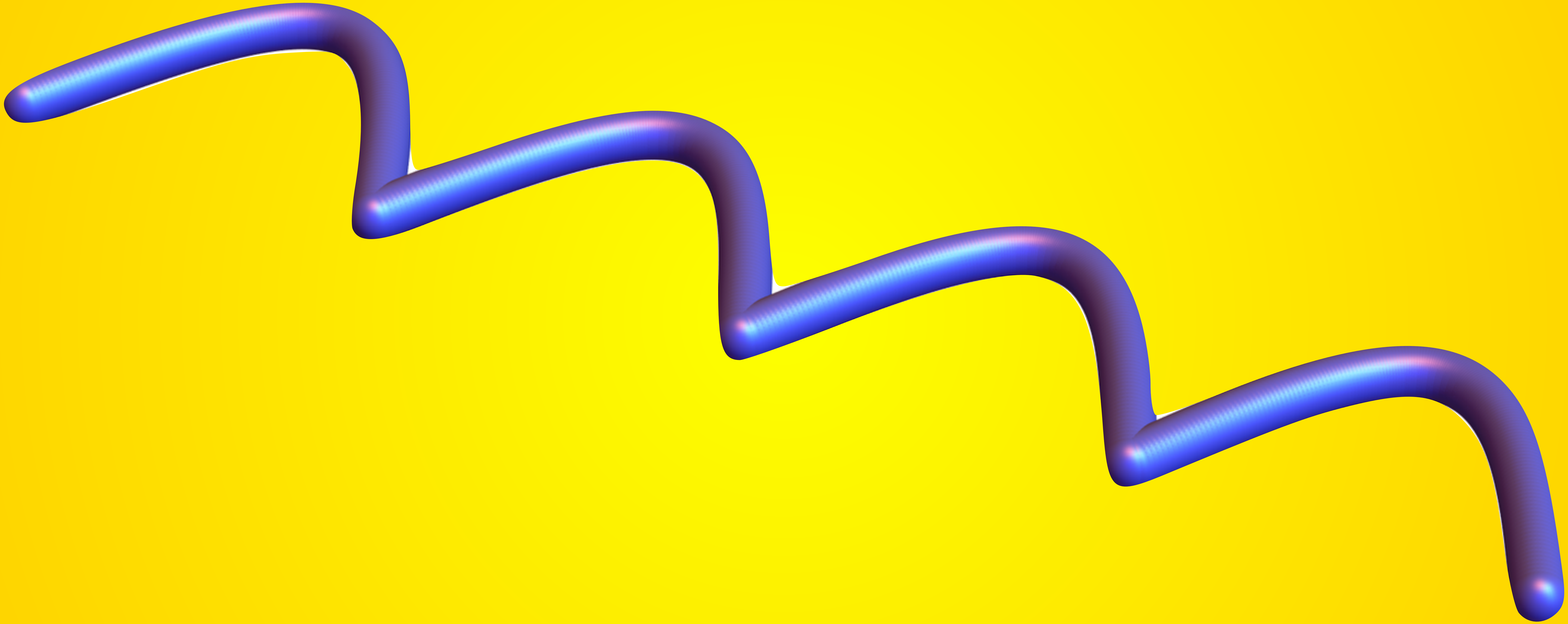
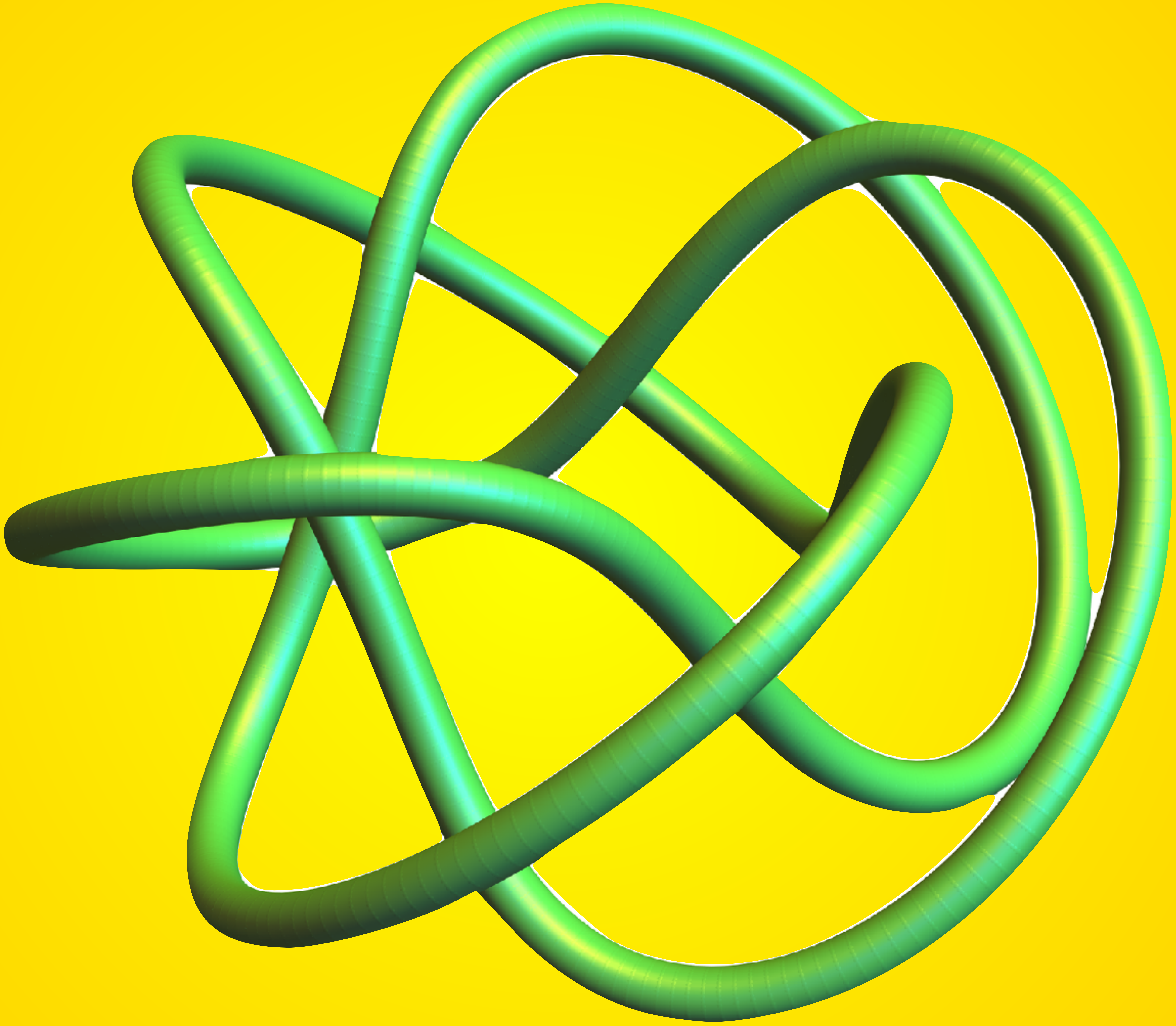


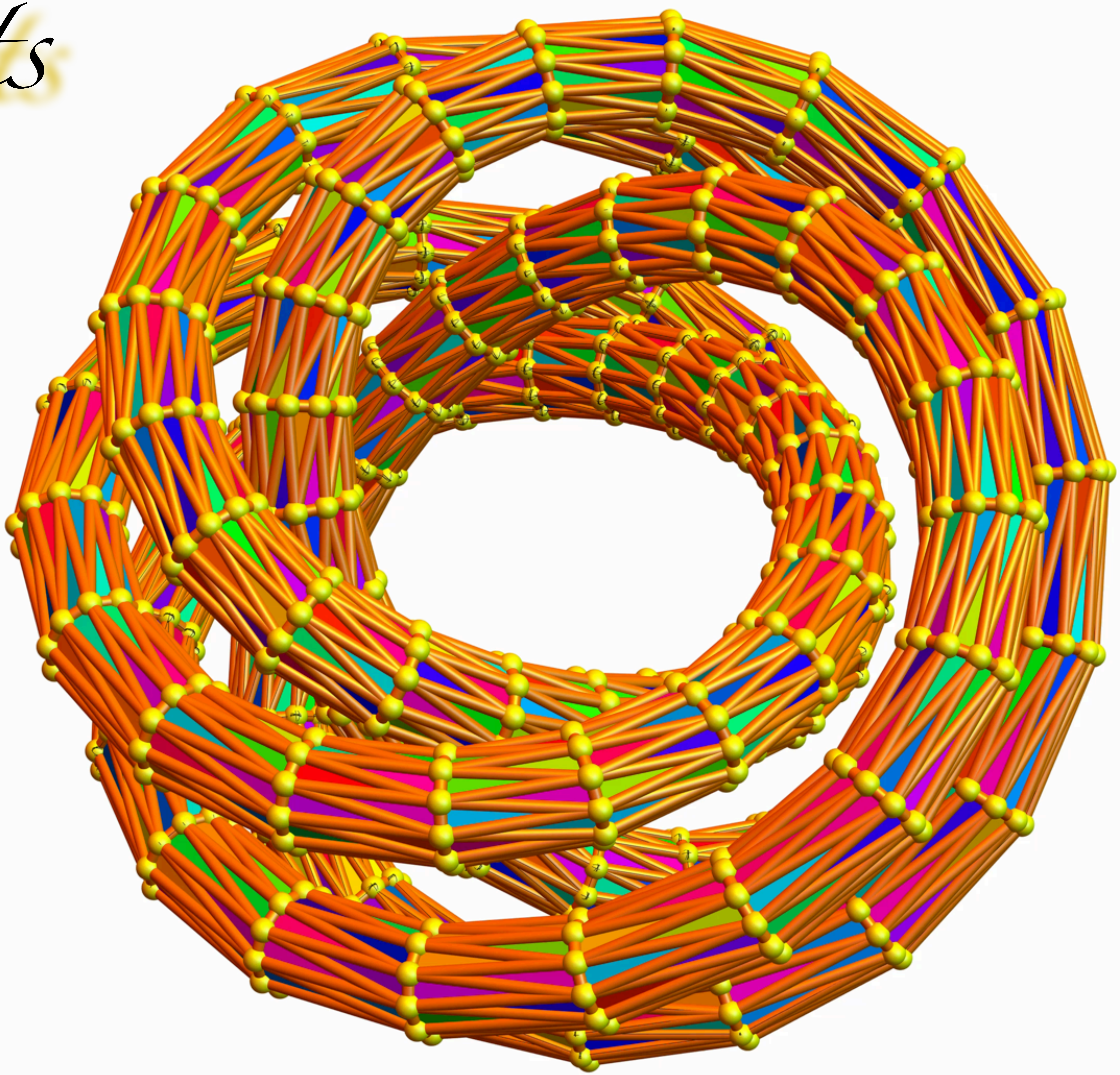






Apropos Curves

knots



Peano Curve

Brownian motion

Dartmouth Electron Microscope Facility, Dartmouth College

youtube

Einstein

Zeno

Democritus

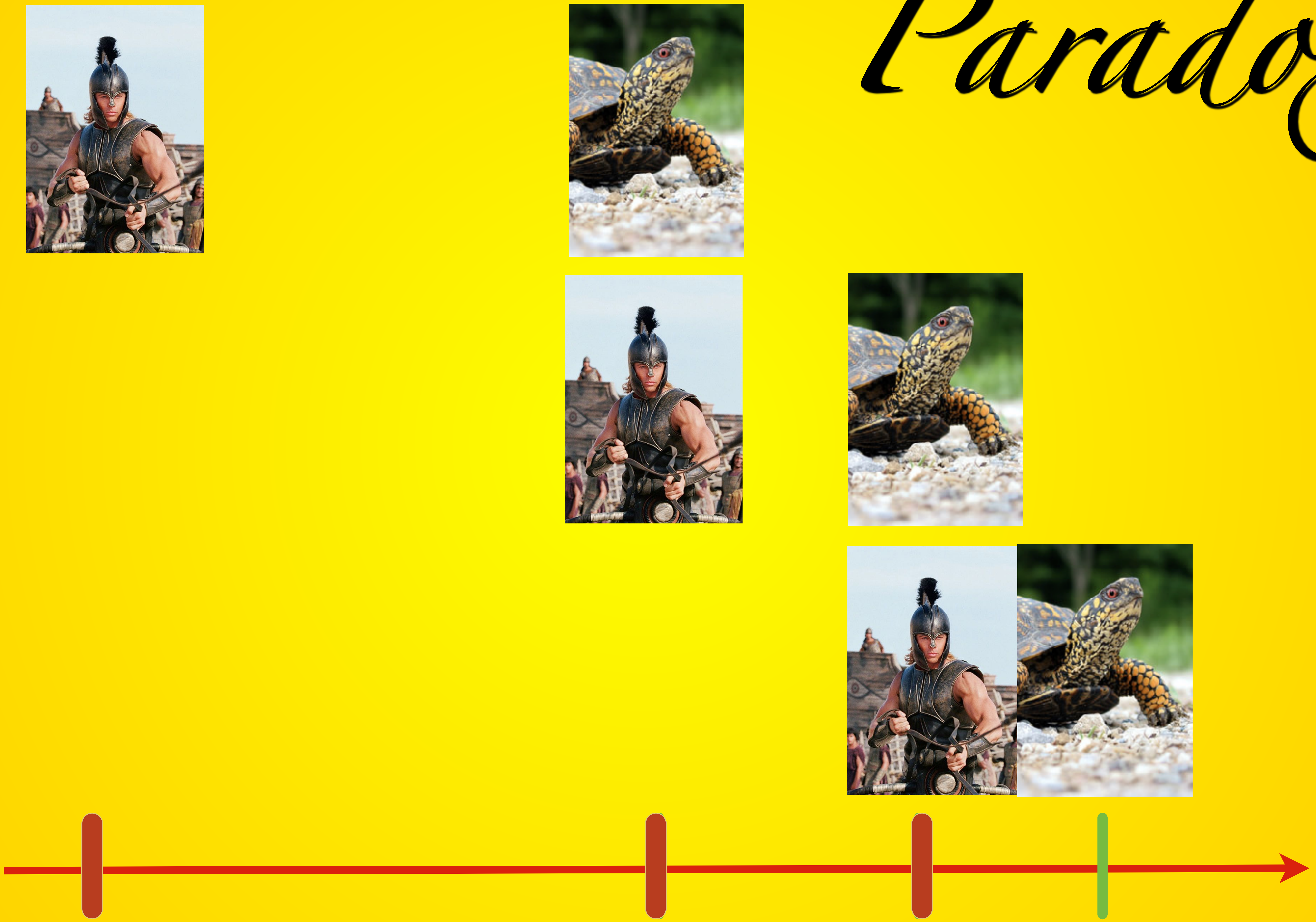
Zeno of Elea

490 – c. 430 BC

Fresco in the Library of El Escorial, Madrid.

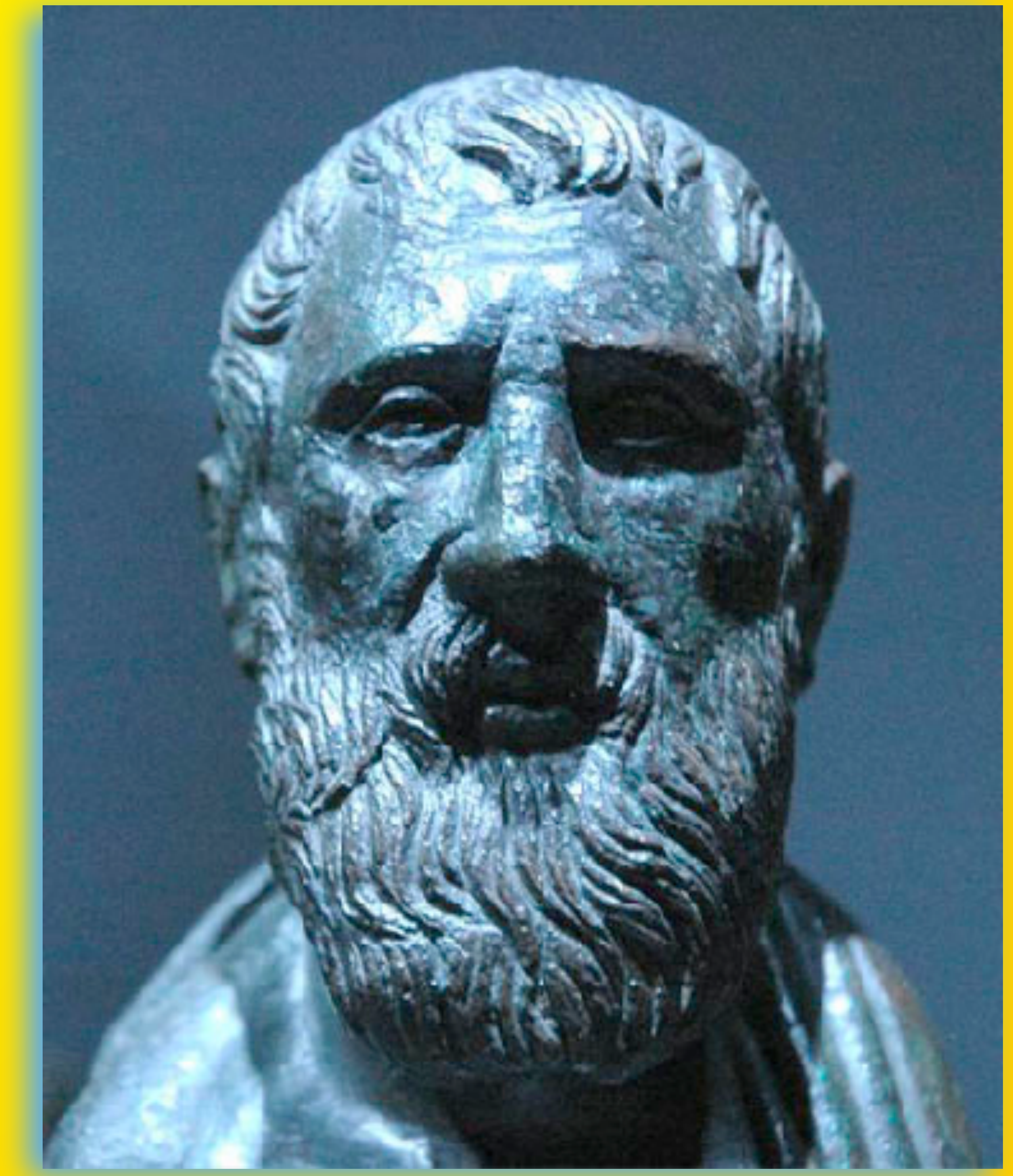
Zeno Paradox

Paradox

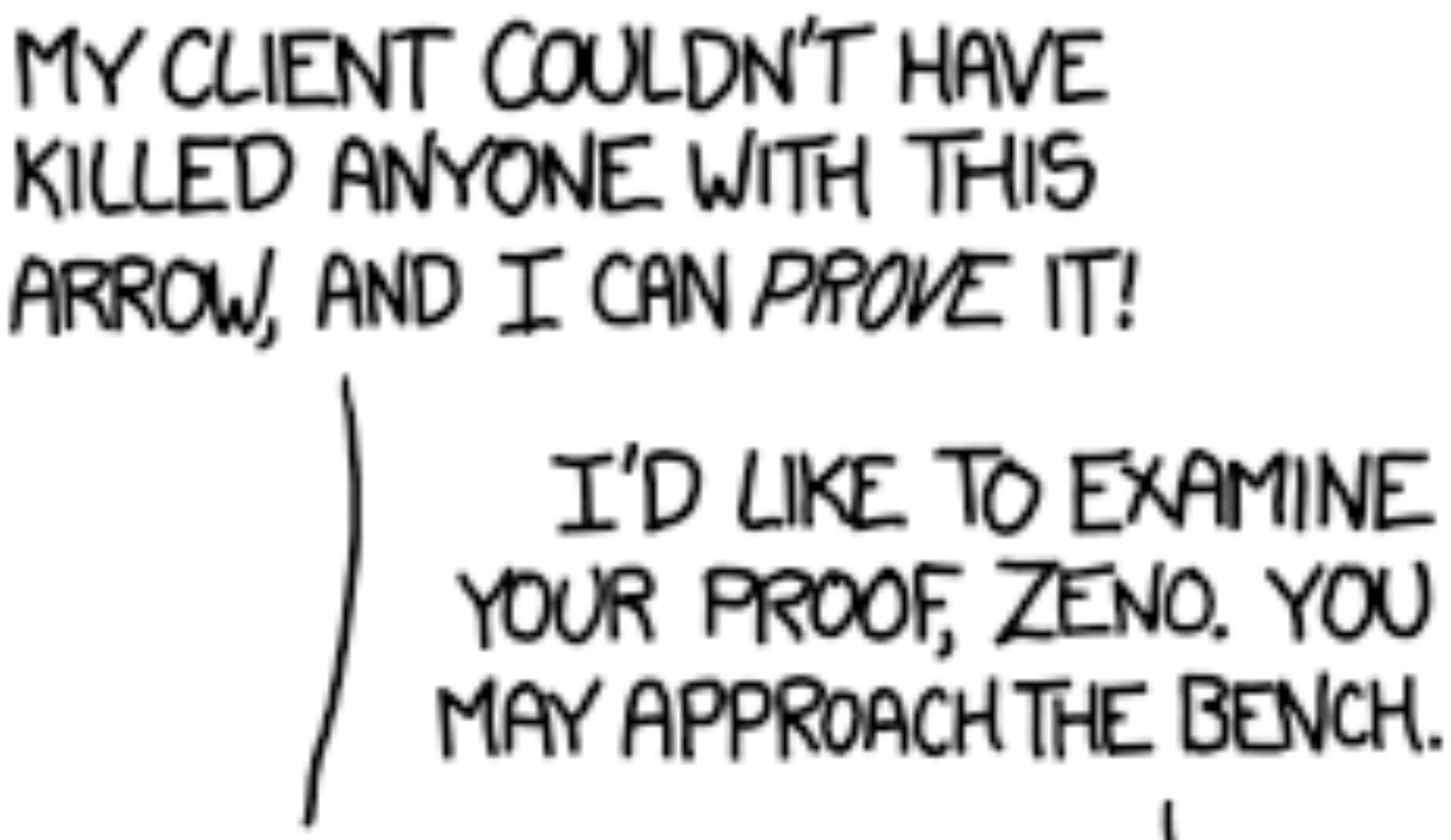


Arrow Paradox

If everything when it occupies an equal space is at rest, and if that which is in locomotion is always occupying such a space at any moment, the flying arrow is therefore motionless.



Zeno ca. 490 - 430 BC


MY CLIENT COULDN'T HAVE KILLED ANYONE WITH THIS ARROW, AND I CAN *PROVE* IT!
I'D LIKE TO EXAMINE YOUR PROOF, ZENO. YOU MAY APPROACH THE BENCH.

The Letter S

The Letter S

Donald E. Knuth

SEVERAL YEARS AGO when I began to look at the problem of designing suitable alphabets for use with modern printing equipment, I found that 25 of the letters were comparatively easy to deal with. The other letter was 'S'. For three days and nights I had a terrible time trying to understand how a proper 'S' could really be defined. The solution I finally came up with turned out to involve some interesting mathematics, and I believe that students of calculus and analytic geometry may enjoy looking into the question as I did. The purpose of this paper is to explain what I now consider to be the 'right' mathematics underlying printed S's, and also to give an example of the METAFONT language I have recently been developing. (A complete description of METAFONT, which is a computer system and language intended to aid in the design of letter shapes, appears in [3, part 3].

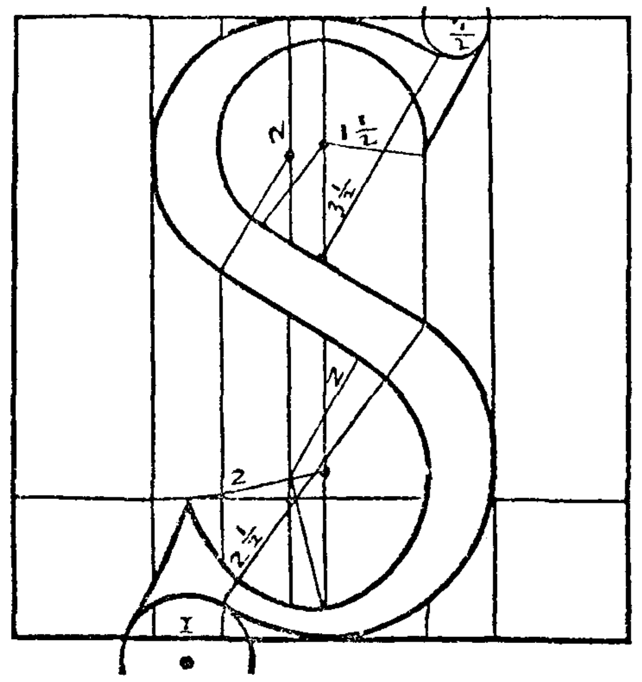
Before getting into a technical discussion, I should probably mention why I started worrying about such things in the first place. The central reason is that today's printing technology is essentially based on discrete mathematics and computer science, not on properties of metals or of movable type. The task of making a plate for a printed page is now essentially that of constructing a gigantic matrix of 0's and 1's, where the 0's specify white space and the 1's specify ink. I wanted the second edition of one of my books to look like the first edition, although the first edition had been typeset with the old hot-lead technology; and when I realized that this problem could be solved by using appropriate techniques of discrete mathematics and computer science, I couldn't resist trying to find my own solution.

Reference [2] explains more of the background of my work, and it also discusses the early history of mathematical approaches to type design. In particular, it illustrates how several people proposed to construct S's geometrically with ruler and compass during the sixteenth and seventeenth centuries.

Francesco Torniello published a geometric alphabet in 1517 that is typical of these early approaches. Let's look at his construction of an 'S' (cf. Fig. 1),

The preparation of this article was supported in part by National Science Foundation grants MCS-7723738 and IST-7921977, by Office of Naval Research grant N00014-76-C-0330, and by the IBM Corporation. The author gratefully acknowledges the help of Xerox Palo Alto Research Laboratory facilities for the preparation of several illustrations. All of the letters and symbols in this report were designed mathematically, using METAFONT.

.1., con quello tondo quale ha lo suo puncto de mezo fora del quadro, longe da la inferiore linea del quadro puncto mezo. Poi largo lo circino puncti .2., ponendo una puncta dove finisti la inferiore parte del .S. qual fu facta a drita linea, cioè longe da la linea del spacio da parte drita puncti .2., e altri



puncti .4. da la [linea] inferiore del quadro. L'altra puncta longe da quella del spacio da parte sinistra puncti .2. descenderai in tondo verso man drita tanto che giongi sopra la media linea. Poi con dicta largheza de circino ponendo l'una puncta dove al presente finisti, l'altra puncta longe da la linea del spacio da parte sinistra puncti .2., venendo dal dicto ultimo loco del .S. tanto che sia lontano da la inferiore linea del quadro puncti .2. Poi da questa ultima parte in tondo vengasi a drita linea a congiungere con lo inferiore tondo longe da la linea da parte sinistra del quadro puncti .1. e sette octavi; & sarà finita la littera .S., come apertamente si vede.

Fig. 1. Francesco Torniello's method of "squaring the S" in 1517. (This is page 45 of [4], reproduced by kind permission of Officina Bodoni in Verona, Italy.)

in order to get some feeling for the problems involved. Paraphrasing his words into modern mathematical terminology, we can state the method as follows:

> An 'S' is drawn in a 9×9 square that we can represent by Cartesian coordinates (x, y) for $0 \le x \le 9$ and $0 \le y \le 9$. We shall define fourteen points on the boundary of the letter, calling them (x_1, y_1), (x_2, y_2), …, (x_{14}, y_{14}). Point 1 is $(4.5, 9)$, and a circular arc is drawn from this point with center at $(4.5, 5.5)$ and radius 3.5 ending at point 2 where $x_2 = 6$.

our previous discussion: We know a point that is supposed to be the top of the S curve, and we also know how far the curve should extend to the left; furthermore we have a straight line in mind that will form the middle link of the stroke.

The problem stated in the preceding paragraph is interesting to me for several reasons. In the first place, it has a nice answer (as we will see). In the second place, the answer does in fact lead to satisfactory S curves. In the third place, the answer isn't completely trivial; during a period of two years or so I came across this problem four different times and each time I was unable to find my notes about how to solve it, so I spent several hours deriving and rederiving the formulas whenever I needed them. Finally I decided to write this paper so that I wouldn't have to derive the answer again.

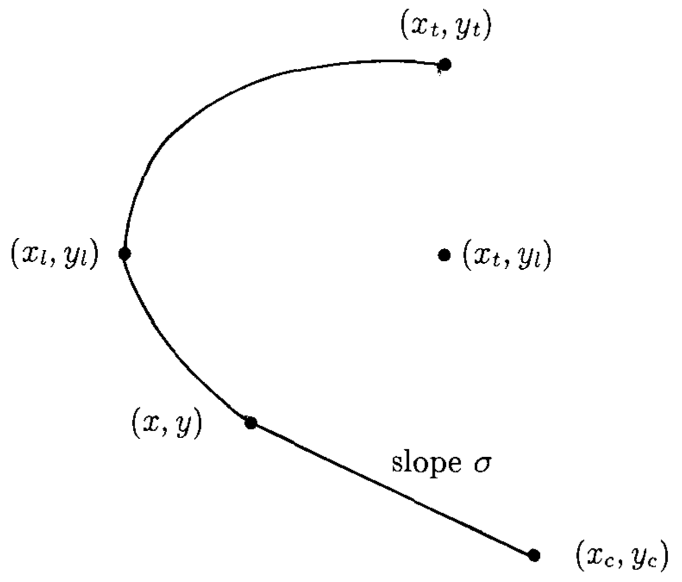


Fig. 7. Problem: Find x, y, and y_l when x_t, y_t, x_l, σ, x_c, and y_c are given.

The point (x_t, y_l) is the center of the ellipse we seek. Let (x, y) be the point where the desired ellipse is tangent to the line of slope σ through (x_c, y_c), as shown in Fig. 7. Our problem boils down to solving three equations in the three unknowns x, y, and y_l:

$$\left(\frac{x - x_t}{x_l - x_t}\right)^2 + \left(\frac{y - y_l}{y_t - y_l}\right)^2 = 1;$$
$$\frac{y_c - y}{x_c - x} = \sigma; \qquad (*)$$
$$-\left(\frac{y_t - y_l}{x_l - x_t}\right)^2 \frac{x - x_t}{y - y_l} = \sigma.$$

The first of these is the standard equation for an ellipse, and the second is the standard equation for a line; the third is obtained by differentiating the first,

$$2\,dx\,\frac{x - x_t}{(x_l - x_t)^2} + 2\,dy\,\frac{y - y_l}{(y_t - y_l)^2} = 0,$$

and setting dy/dx equal to σ.

Before attempting to solve equations $(*)$, I would like to introduce a notation that has turned out to be extremely useful in my work on mathematical font design: Let $\alpha[x, y]$ be an abbreviation for

$$x + \alpha(y - x),$$

which may be understood as "the fraction α of the way from x to y". Thus $0[x, y] = x$; $1[x, y] = y$; $\frac{1}{2}[x, y]$ is the midpoint between x and y; $\frac{3}{4}[x, y]$ is halfway between y and this midpoint; and $2[x, y]$ lies on the opposite side of y from x, at the same distance as y is from x. Identities like $\alpha[x, x] = x$ and $\alpha[x, y] = (1 - \alpha)[y, x]$ are easily derived. When making some geometric construction it is common to refer to things like the point one third of the way from A to B; the notation $\frac{1}{3}[A, B]$ means just that.

One of the uses of this bracket notation is to find the intersection (x, y) of two given lines, where the lines go respectively from (x_1, y_1) to (x_2, y_2) and from (x_3, y_3) to (x_4, y_4). We can solve the intersection problem by noting that there is some number α such that

$$x = \alpha[x_1, x_2], \qquad y = \alpha[y_1, y_2]$$

and some number β such that

$$x = \beta[x_3, x_4], \qquad y = \beta[y_3, y_4].$$

These four simultaneous linear equations in x, y, α, β are easily solved; and in fact METAFONT will automatically solve simultaneous linear equations, so it is easy to compute the intersection of lines in METAFONT programs.

The bracket notation also applies to ellipses in an interesting way. We can write $x = \alpha[x_0, x_{\max}]$ and $y = \beta[y_0, y_{\max}]$ in the general equation

$$\left(\frac{x - x_0}{x_{\max} - x_0}\right)^2 + \left(\frac{y - y_0}{y_{\max} - y_0}\right)^2 = 1,$$

reducing it to the much simpler equation

$$\alpha^2 + \beta^2 = 1.$$

Returning to our problem of the ellipse, let us set

$$x = \alpha[x_t, x_l], \qquad y = \beta[y_l, y_t],$$
$$X = x - x_t, \qquad Y = y_l - y_t,$$
$$a = x_l - x_t, \qquad b = (y_c - \sigma x_c) - (y_t - \sigma x_t).$$

The three equations $(*)$ can now be rewritten as follows:

$$\alpha^2 + \beta^2 = 1;$$
$$b + \sigma X = (1 - \beta)Y; \qquad (**)$$
$$aY = a\sigma\beta;$$
$$X = a\alpha.$$

This gives us four equations in the four unknowns (α, β, X, Y), so it may seem that we have taken a step

THE END