INTRODUCTION TO CALCULUS

MATH 1A

Unit 35: Review

The older review sheets are contained in unit 15 and unit 28. We do here not recopy those parts.

Substitution

Substitution replaces $\int f(x)\,dx$ with $\int g(u)\,du$ with $u = u(x), du = u'(x)dx$. Cases:

A) The integral of $f(x) = g(u(x))u'(x)$, is $G(u(x))$ where G is the anti derivative of g.
B) $\int f(ax+b)\,dx = F(ax+b)/a$ where F is the anti derivative of f.

Examples:
A) $\int \sin(x^5)x^4\,dx = \int \sin(u)\,du/5 = -\cos(u)/5 + C = -\cos(x^5)/5 + C$.
B) $\int \log(5x+7)\,dx = \int \log(u)\,du/5 = (u\log(u) - u)/5 + C = (5x+7)\log(5x+7) - (5x+7) + C$.

Integration by parts

A) Direct (use this if it can be done in one step)

$$\int \boxed{\mathrm{x}}\ \boxed{\sin(x)}\ dx = \boxed{\mathrm{x}}\ \boxed{(-\cos(x))} - \int \boxed{1}\ \boxed{(-\cos(x))}\ dx = -x\cos(x) + \sin(x) + C\ dx\ .$$

B) Tic-Tac-Toe: to integrate $x^2\sin(x)$:

x^2	$\sin(x)$	
$2x$	$-\cos(x)$	$\oplus$
2	$-\sin(x)$	$\ominus$
0	$\cos(x)$	$\oplus$

The anti-derivative is

$$-x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C\ .$$

C) Merry go round: Example $I = \int \sin(x)e^x\,dx$. Use parts twice and solve for I.

Partial fractions

A) Make a common denominator on the right hand side $\frac{1}{(x-a)(x-b)} = \frac{A(x-b)+B(x-a)}{(x-a)(x-b)}$. and compare coefficients $1 = Ax - Ab + Bx - Ba$ to get $A + B = 0, Ab - Ba = 1$ and solve for A, B.
B) The residue method works better if $f(x) = p(x)/(x-a)(x-b)$ with different a, b, the coefficients A, B in $\frac{p(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$ can be obtained from

$$A = \lim_{x \to a}(x-a)f(x) = p(a)/(a-b),\ B = \lim_{x \to b}(x-b)f(x) = p(b)/(b-a)\ .$$

Examples:
A) $\int \frac{1}{(x+1)(x+2)}\,dx = \int \frac{A}{x+1}\,dx + \int \frac{B}{x+2}\,dx$. Find A, B by multiplying out and comparing coefficients in the nominator.
B) Directly write down $A = 1$ and $B = -1$, by plugging in $x = -2$ after multiplying with $x - 2$. or plugging in $x = -1$ after multiplying with $x - 1$.

Improper integrals

A) Integrate over infinite domain.
B) Integrate over point where f is unbounded.

Examples:
A) $\int_0^\infty 1/(1+x^2) = \arctan(\infty) - \arctan(0) = \pi/2 - 0 = \pi/2$.
B) $\int_0^1 1/x^{2/3}\,dx = (3/1)x^{1/3}|_0^1 = 3$.

Trig substitution

When integrating function like $\sqrt{1-x^2}$, replace x by $\sin(u)$. **Example:**
$\int_{-1}^1 \sqrt{1-x^2}\,dx = \int_{-\pi/2}^{\pi/2} \cos(u)\cos(u)du = \int_{-\pi/2}^{\pi/2}(1+\cos(2u))/2 = \frac{\pi}{2}$.

Terminology in Application

Music: hull function, piano function
Economics: average cost, marginal cost and total cost. Strawberry theorem
Operations research: find extrema, critical points, derivative tests
Computer science: random function from given function
Statistics: PDF, cumulative distribution function, expectation, variance.
Distributions: normal distribution, geometric distribution, Cauchy distribution.
Geometry: area between two curves, volume of solid
Numerical integration: Riemann sum, trapezoid rule, Simpson rule, Monte Carlo
Root finding: Bisection method, Newton method $T(x) = x - f(x)/f'(x)$.
Psychology: critical points and catastrophes.

Physics: position, velocity and acceleration, work and power.
Gastronomy: turn table to prevent wobbling, bottle calibration.

Checklists:

Integral techniques to consider

Try in the following order:

- ☐ Know the integral
- ☐ Substitution
- ☐ Trig substitution
- ☐ Integration by parts
- ☐ Partial fractions

Especially:

- ☐ Tic-Tac-Toe for integration by parts
- ☐ Hospital Method for partial fractions
- ☐ Merry go round method for parts

Integrals to know well

- ☐ $\sin(x)$
- ☐ $\cos(x)$
- ☐ $\tan(x)$
- ☐ $\log(x)$
- ☐ $\exp(x)$
- ☐ $1/x$
- ☐ $1/x^n$
- ☐ x^n
- ☐ $\sqrt{x}$
- ☐ $1/\cos^2(x)$
- ☐ $1/\sin^2(x)$
- ☐ $1/(1+x^2)$
- ☐ $1/(1-x^2)$
- ☐ $1/\sqrt{1-x^2}$
- ☐ $\sqrt{1-x^2}$

Applications you have to know

- ☐ **Derivative:** Limit of differences $D_h f = [f(x+h) - f(x)]/h$ for $h \to 0$
- ☐ **Integral:** Limit of Riemann sums $S_h f = [f(0) + f(h) + \cdots + f(kh)]h$.
- ☐ **Newton step:** $T(x) = x - f(x)/f'(x)$.
- ☐ **Marginal cost:** the derivative F' of the total cost F.
- ☐ **Average cost:** F/x where F is the total cost.
- ☐ **Velocity:** Derivative of the position.
- ☐ **Acceleration:** Derivative of the velocity.
- ☐ **Curvature:** $f''(x)/(1 + f'(x)^2)^{3/2}$.
- ☐ **Probability distribution function:** non-negative function with total $\int f(x)dx = 1$.
- ☐ **Cumulative distribution function:** anti-derivative of the PDF.
- ☐ **Expectation:** $\int xf(x)\,dx$, where f is the probability density function.
- ☐ **Piano function:** frequencies $f(k) = 440 \cdot 2^{k/12}$ for integer k.
- ☐ **Hull function:** $\sin(x)\sin(10000x)$ has hull $|\sin(x)|$
- ☐ **Catastrophe:** A parameter c at which a local minimum disappears.

Core concepts

- ☐ **Fundamental:** The fundamental theorem of calculus
- ☐ **Extrema:** Second derivative test
- ☐ **Derivatives:** slope rate of change
- ☐ **Integrals:** area, volume
- ☐ **Limits:** Hospital!
- ☐ **Continuity:** know the enemies of continuity
- ☐ **Numerics:** Riemann sum, Trapezoid and Simpson rule
- ☐ **Rules:** Differentiation and integration rules.
- ☐ **Methods:** Integration by parts, Substitution, Partial fraction.

Not needed on your fingertips but fair game

- ☐ **Epidemic:** logistic growth.
- ☐ **Entropy:** $-\int f(x)\log(f(x))\,dx$.
- ☐ **Monte Carlo integration:** $S_n = \frac{1}{n}\sum_{k=1}^{n} f(x_k)$, where x_k are random in $[a, b]$.
- ☐ **Bart Simpson rule:** $S_n = \frac{1}{6n}\sum_{k=1}^{n}[f(x_k) + 4f(y_k) + f(x_{k+1})]$.
- ☐ **Cocktail party stuff:** Eat,integrate and love.
- ☐ **Bottles:** How to calibrate bottles. The calibration formula.
- ☐ **Sofia:** The name of a calculus bot once living in the math department.
- ☐ **Wobbly chair:** One can turn a chair on any lawn to stop it from wobbling.
- ☐ **Song:** The hit: "low d high take high d low, cross the line and square the low"
- ☐ **1a Song:** The golden record: "Good bye Math 1A, this is the end of the season"
- ☐ **Midi function:** $f(m) = 440 * 2^{(m-69)/12}$.