INTRODUCTION TO CALCULUS

MATH 1A

Unit 31: Calculus and Economics

Lecture

31.1. Calculus plays a pivotal role in **economics**. This unit is an opportunity to review extrema problems and get acquainted with some jargon in economics. Economists talk differently: $f' > 0$ means **growth** or **boom**, $f' < 0$ means **decline** or **recession**, a vertical asymptote is a **crash**, a horizontal asymptote is a **stagnation**, a discontinuity is **"inelastic behavior"**, the derivative of something is the **"marginal"** of it. An example is marginal revenue.

Definition: The **marginal cost** is the derivative of the **total cost**.

31.2. The marginal cost and total cost are functions of the quantity x of goods:

Example: Assume the total cost function is $C(x) = 10x - 0.01x^2$. Use the marginal cost in order to minimize the total cost. **Solution**. Differentiate $C' = 10 + 0.02x$ At a minimum, this derivative is zero. Here at $x = 50$.

Example: You sell spring water. The marginal cost to produce it at time x (years) is $f(x) = 1000 - 2000\sin(x/6)$. For which x is the total cost maximal? **Solution**. We look for points where $F'(x) = f(x) = 0, F''(x) < 0$. This is the case for $x = \pi$. The cost is maximal in about 3 years.

Example: In the book "Don't worry about Micro, 2008", by Dominik Heckner and Tobias Kretschmer, the following strawberry story appears: (italics is verbatim):

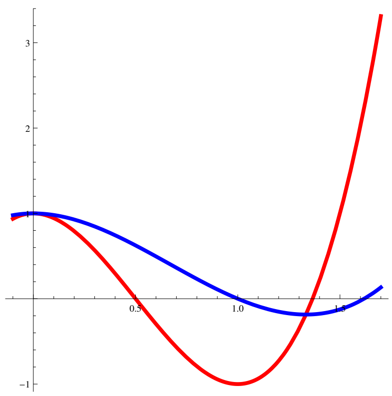
Suppose you have all sizes of strawberries, from very large to very small. Each size of strawberry exists twice except for the smallest, of which you only have one. Let us also say that you line these strawberries up from very large to very small, then to very large again. You take one strawberry after another and place them on a scale that sells you the average weight of all strawberries. The first strawberry that you place in the bucket is very large, while every subsequent one will be smaller until you reach the smallest one. Because of the literal weight of the heavier ones, average weight is larger than marginal weight. Average weight still decreases, although less steeply than marginal weight. Once you reach the smallest strawberry, every subsequent strawberry will be larger which means that the rate of decease of the average weight becomes smaller and smaller until eventually, it stands still. At this point the marginal weight is just equal to the average weight.

31.3. If $F(x)$ is the **total cost function** in dependence of the quantity x, then $F' = f$ is called the **marginal cost**.

Definition: The function $g(x) = F(x)/x$ is called the **average cost**.

Definition: A point where $f = g$ is called a **break-even point**.

Example: If $f(x) = 4x^3 - 3x^2 + 1$, then $F(x) = x^4 - x^3 + x$ and $g(x) = x^3 - x^2 + 1$. Find the break even point and the points, where the average costs are extremal. **Solution:** To get the break even point, we solve $f - g = 0$. We get $f - g = x^2(3x - 4)$ and see that $x = 0$ and $x = 4/3$ are two break even points. The critical point of g are points where $g'(x) = 3x^2 - 4x$. They agree:



31.4. The following theorem tells that the marginal cost is equal to the average cost if and only if the average cost has a critical point. Since total costs are typically concave up, we usually have "break even points are minima for the average cost". Since the strawberry story illustrates it well, lets call it the "strawberry theorem":

Strawberry theorem: $g'(x) = 0$ if an only if $f = g$.

Proof.

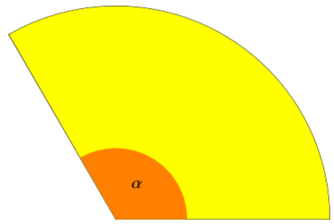
$$g' = (F(x)/x)' = F'/x - F/x^2 = (1/x)(F' - F/x) = (1/x)(f - g) .$$

More extremization

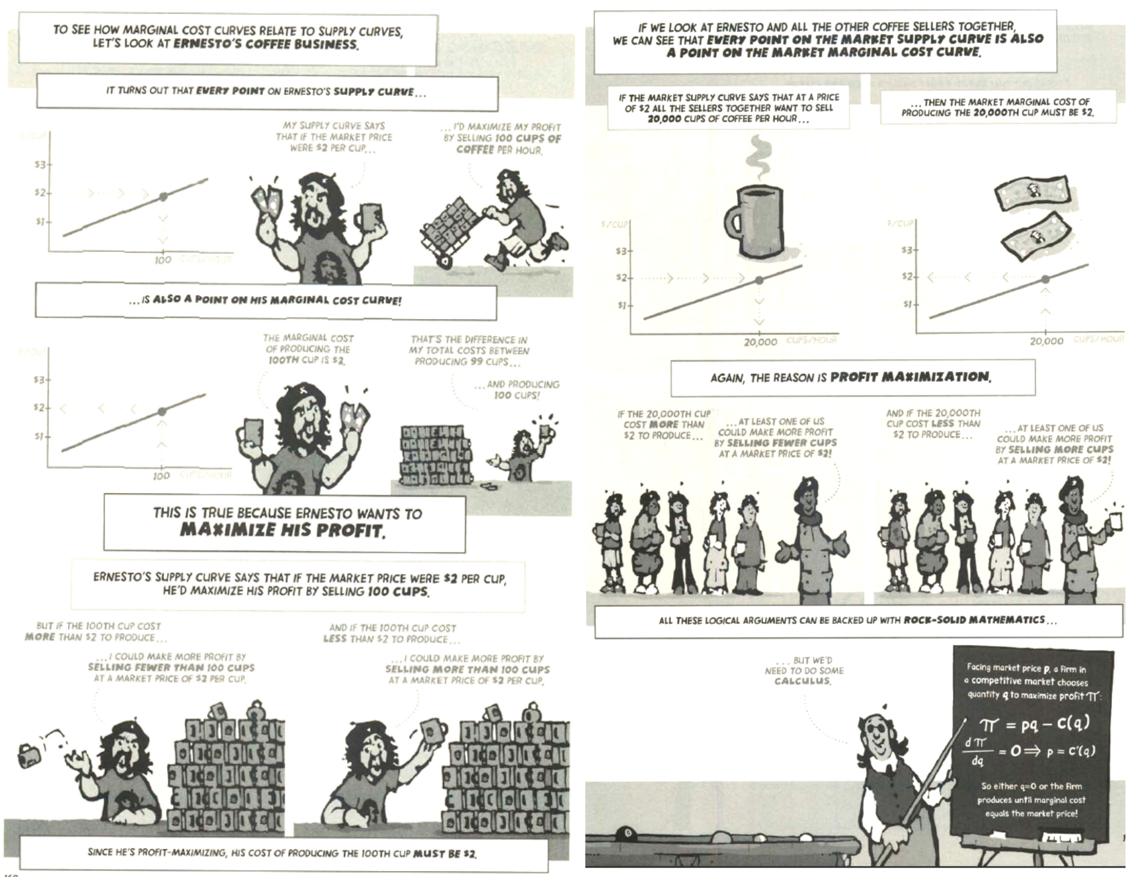
31.5. In the second part of this lecture we still want to look at more extremization problems, also in the context of data.

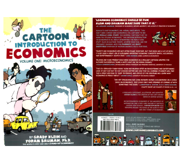
Example: Find the **rhomboid** with side length 1 which has maximal area. Use angle α to extremize. Solution. The area is $base * height = 1 * \sin(\alpha)$. This is maximal at $\pi/2$.

Example: Find the sector of radius$r = 1$ and angle α which has minimal circumference $f = 2r + r\alpha$ if the area $r^2\alpha/2 = 1$ is fixed. **Solution.** Find $\alpha = 2/r^2$ from the second equation and plug it into the first equation. We get $f(r) = 2r + 2/r$. Now the task is to find the places where $f'(r) = 0$.



Example: Find the ellipse of length $2a$ and width $2b$ which has fixed area $\pi ab = \pi$ and for which the sum of diameters $2a + 2b$ is maximal. **Solution.** Find $b = 1/a$ from the first equation and plug into the second equation. Again we have to extremize $f(a) = 2a + 2/a$ as before.





Source: Grady Klein and Yoram Bauman, The Cartoon Introduction to Economics: Volume One Microeconomics, published by Hill and Wang. You can detect the strawberry theorem ($g' = 0$ is equivalent to $f = g$) can be seen on the blackboard.

Homework

Problem 31.1: a) Find the break-even point for an economic system if the marginal cost is $f(x) = 1/x$.
b) Assume the marginal cost is $f(x) = x^7$. Verify that the average cost $g(x) = F(x)/x$ satisfies $8g(x) = f(x)$ and that $x = 0$ is the only break even point.

Problem 31.2: Let $f(x) = \cos(x)$. Compute $F(x)$ and $g(x)$ and verify that $f = g$ agrees with $g' = 0$.

Problem 31.3: For smaller groups, production usually increases when adding more workforce. After some time, bottle necks occur, not all resources can be used at the same management and bureaucracy is added. We make a model to find the maximal production parameters. The **production function** in an office gives the production $Q(L)$ in dependence of labor L. Assume

$$Q(L) = 5000L^3 - 3L^5 .$$

Find L which gives the maximal production.

Problem 31.4: **Marginal revenue** f is the rate of change in **total revenue** F. As total and marginal cost, these are functions of the **cost** x. Assume the total revenue is $F(x) = -5x - x^5 + 9x^3$. Find the points, where the total revenue has a local maximum.

Problem 31.5: We do linear regression. Find the best line $y = mx + b$ through the points

$$(x_1, y_1) = (1, 3), (x_2, y_2) = (2, 6), (x_3, y_3) = (3, 6) .$$

To do so, first center the data by subtracting the average (2,5), then minimize $f(m) = \sum_{k=1}^{3}(mx_k - y_k)^2$ Now $m(x-2)+5$ is the best fit.