INTRODUCTION TO CALCULUS

MATH 1A

Unit 20: Area

Lecture

20.1. If $f(x) \geq 0$, then $\int_a^b f(x)\ dx$ is the **area under the graph** of $f(x)$ and above the interval $[a, b]$ on the x-axis. If the function is negative, then $\int_a^b f(x)\ dx$ is negative too and the integral is minus the area below the curve:

Therefore, $\int_a^b f(x)\ dx$ is the difference of the area above the graph minus the area below the graph. We call it a **signed area**.

20.2. More generally we can also look at **areas sandwiched between two graphs** f **and** g.

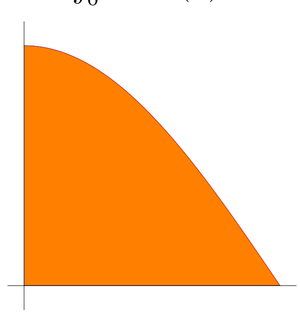
The area of a region G enclosed by two graphs $f \leq g$ and bound by $a \leq x \leq b$ is

$$\int_a^b g(x) - f(x)\ dx$$

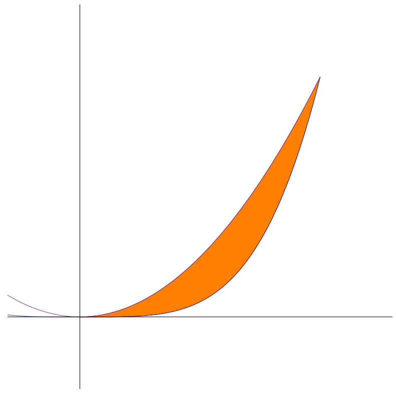
.

20.3. Make sure that if you have to compute such an integral that $g \geq f$ before giving it the interpretation of an area.

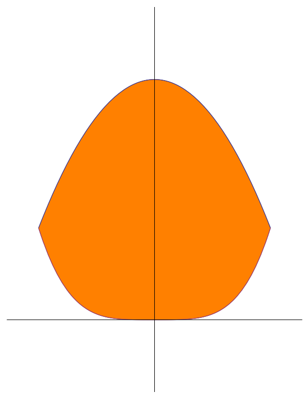
Example: Find the area of the region bound by the cos function and the x and y axes. cos function. **Solution**: $\int_0^{\pi/2} \cos(x)\ dx = 1$.



Example: Find the area of the region enclosed by the graphs $f(x) = x^2$ and $f(x) = x^4$.



Example: Find the area of the region enclosed by the graphs $f(x) = 1 - x^2$ and $g(x) = x^4$. **Solution**: The intersection points are $\pm(\sqrt{5}-1)/2$ and called golden ratio. Now it is routine.

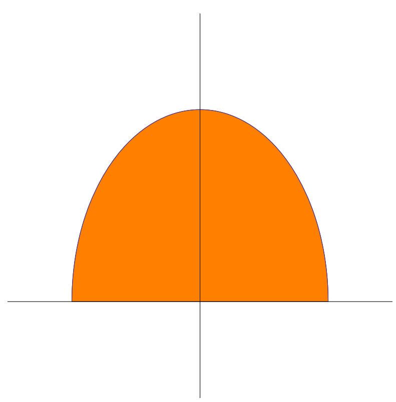


Example: Find the area of the region enclosed by a half circle of radius 1. **Solution**: The half circle is the graph of the function $f(x) = \sqrt{1-x^2}$. The area under the graph is

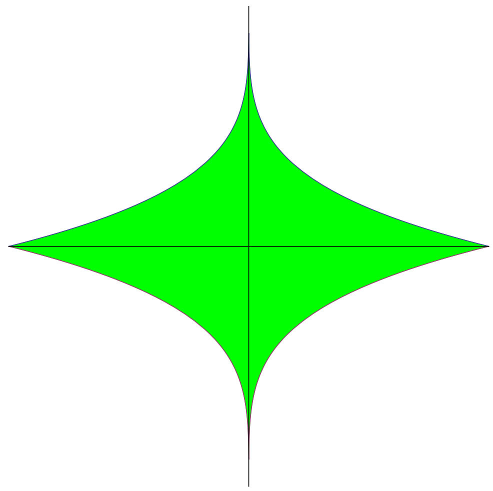
$$\int_{-1}^{1} \sqrt{1-x^2}\, dx \ .$$

Finding the anti-derivative is not so easy. We will find techniques to do so, for now we just are told to look at the derivative of $x\sqrt{1-x^2} + \arcsin(x)$ and see what happens. With this "inspiration", we find the anti derivative to be $(x\sqrt{1-x^2} + \arcsin(x))/2$. The area is therefore

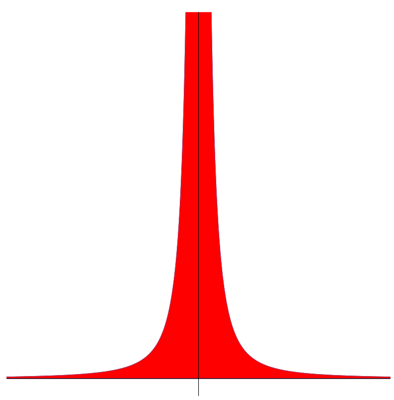
$$\frac{x\sqrt{1-x^2} + \arcsin(x)}{2}\Big|_{-1}^{1} = \frac{\pi}{2} \ .$$



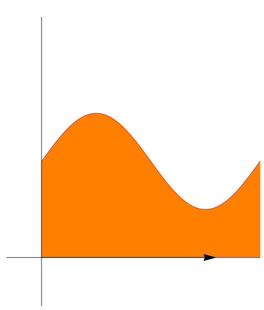
Example: Find the area of the region between the graphs of $f(x) = 1 - |x|^{1/4}$ and $g(x) = -1 + |x|^{1/4}$.



Example: Find the area under the curve of $f(x) = 1/x^2$ between -6 and 6. Naive solution attempt. $\int_{-6}^{6} x^{-2}\, dx = -x^{-1}|_{-6}^{6} = -1/6 - 1/6 = -1/3$. There is something fishy with this computation because $f(x)$ is non-negative so that the area should be positive. But we obtained a negative answer. What is going on?



Example: Find the area between the curves $x = 0$ and $x = 2 + \sin(y)$, $y = 2\pi$ and $y = 0$. **Solution:** We turn the picture by 90 degrees so that we compute the area under the curve $y = 0, y = 2 + \sin(x)$ and $x = 2\pi$ and $x = 0$.



Example: **The grass problem**. Find the area between the curves $|x|^{1/3}$ and $|x|^{1/2}$.
Solution. This example illustrates how important it is to have a picture. This is good advise for any "word problem" in mathematics.

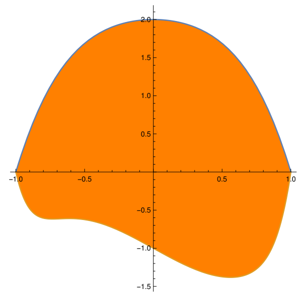
Use a picture of the situation while doing the computation.

Homework

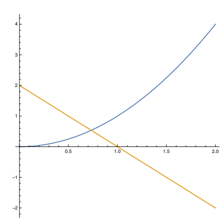
Problem 20.1: Find the area of the bounded region enclosed by the graphs $f(x) = x^4 - 6$ and $g(x) = x^2 + 6$ for $x > 0$. It is a good idea to make a picture.

Problem 20.2: Find the area of the region enclosed by the curves $x = 0, x = \pi/2, y = 4 + \sin(11x), y = \sin^2(2x)$.

Problem 20.3: Find the area of the region enclosed by the graphs $2 - x^4 - x^2$ and $x^{10} - 1 + x^3 - x$.



Problem 20.4: Find the area of the region enclosed by the three curves $y = x^2, y = 2 - 2x$ and $y = 0$.



Problem 20.5: Write down an integral which gives the area of the **area 51** region $x^2 + |y|^{51} \leq 1$ by writing the region as a sandwich between two graphs. Evaluate the integral numerically using Wolfram alpha, Mathematica or any other software.

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