INTRODUCTION TO CALCULUS

MATH 1A

Unit 11: Critical Points

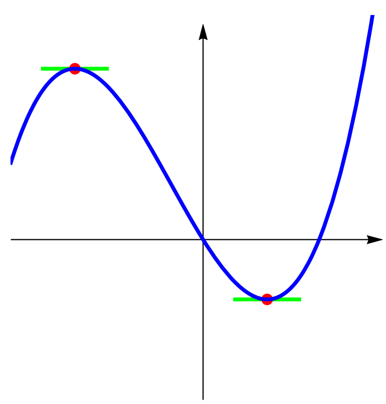
Lecture

11.1. We like to maximize nice quantities and minimize unpleasant ones. Optimizing quantities is also an important principle which nature follows: laws in physics like Newton's law describing the motion of planets, or the Maxwell's equations describing the propagation of light, or the equations written down by Einstein to describe how matter influences geometric space are based on the principle of extremization. An important intuitive insight is that at maxima or minima of a function f, the tangent to the graph must be horizontal. This leads to the following notion for differentiable functions:

Definition: A point x_0 is a **critical point** of f if $f'(x_0) = 0$.

11.2. In some textbooks, critical points also include points, where f' is not defined. Others also include boundary points. [1] We therefore do **not** include boundary points when we make a list of critical points. These points are considered to be outside the domain of definition of f' and we deal with them separately.

Example: Find the critical points of the function $f(x) = x^3 + 3x^2 - 24x$. **Solution:** we compute the derivative as $f'(x) = 3x^2 + 6x - 24$. The roots of f' are $2, -4$.



Definition: A point is called a **local maximum** of f, if there exists an interval $U = (p - a, p + a)$ around p, such that $f(p) \geq f(x)$ for all $x \in U$. A **local minimum** is a local maximum of $-f$. Local maxima and minima together are called **local extrema**.

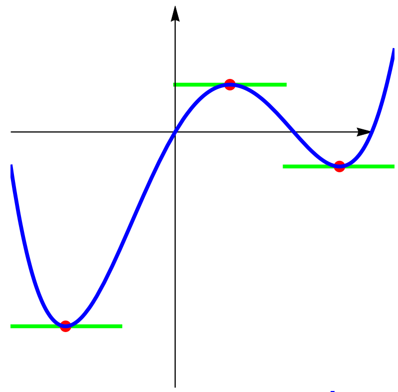
[1] In all more advanced math textbooks, critical points are defined points where $f' = 0$. Important definitions have to be simple.

Example: The point $x = 0$ is a local maximum for $f(x) = \cos(x)$. The reason is that $f(0) = 1$ and $f(x) < 1$ nearby.

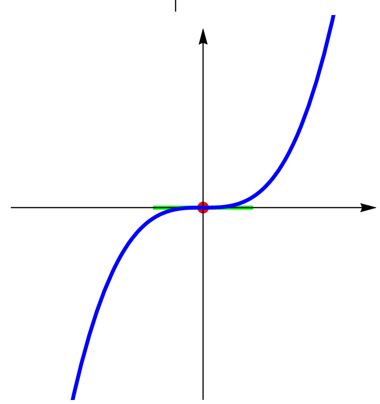
Example: The point $x = 1$ is a local minimum for $f(x) = (x-1)^2$. The function is zero at $x = 1$ and positive everywhere else.

Fermat: If f is differentiable and has a local extremum at x, then $f'(x) = 0$.

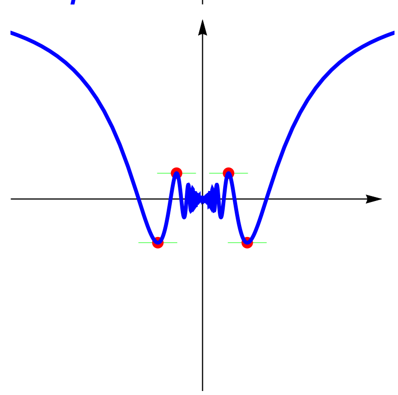
11.3. Why is this so? Assume the derivative $f'(x) = c$ is non-zero. We can assume $c > 0$ otherwise replace f with $-f$. By the definition of limits, for some small enough h, we have $f(x+h) - f(x)/h \geq c/2$. But this means $f(x+h) \geq f(x) + hc/2$ and x can not be a local maximum.



Example: The derivative of $f(x) = 72x - 30x^2 - 8x^3 + 3x^4$ is $f'(x) = 72 - 60x - 24x^2 + 12x^3$. By plugging in integers (calculus teachers like integer roots because students like integer roots!) we can guess the roots $x = 1, x = 3, x = -2$ and see $f'(x) = 12(x-1)(x+2)(x-3)$. The critical points are $1, 3, -2$.



Example: We have already seen that $f'(x) = 0$ does not necessarily imply that x is a local maximum or minimum. The function $f(x) = x^3$ is a counter example. It satisfies $f'(0) = 0$ but 0 is neither a minimum nor maximum there. It is an example of an **inflection point**, which is a point, where the second derivative f'' changes sign.



Example: The function $f(x) = x\sin(1/x)$ is continuous at $x = 0$ but there are infinitely many critical points near 0. The function f is not differentiable at 0, the derivative $\sin(1/x) - \cos(1/x)/x$ not only oscillates like crazy at $x = 0$, it also blows up at $x = 0$.

11.4. If $f''(x) > 0$, then the graph of the function is **concave up**. If $f''(x) < 0$ then the graph of the function is **concave down**.

Second derivative test. If x is a critical point of f and $f''(x) > 0$, then f is a local minimum. If $f''(x) < 0$, then f is a local maximum.

11.5. If $f''(x_0) > 0$ then $f'(x)$ is negative for $x < x_0$ and positive for $f'(x) > x_0$. This means that the function decreases left from the critical point and increases right from the critical point. The point x_0 is a local minimum. Similarly, if $f''(x_0) < 0$ then $f'(x)$ is positive for $x < x_0$ and $f'(x)$ is positive for $x > x_0$. This means that the function increases left from the critical point and increases right from the critical point. The point is a local maximum.

Example: The function $f(x) = x^2$ has one critical point at $x = 0$. Its second derivative is 2 there.

Example: Find the local maxima and minima of the function $f(x) = x^3 - 3x$ using the second derivative test. **Solution:** $f'(x) = 3x^2 - 3$ has the roots $1, -1$. The second derivative $f''(x) = 6x$ is negative at $x = -1$ and positive at $x = 1$. The point $x = -1$ is therefore a local maximum and the point $x = 1$ is a local minimum.

Example: Find the local maxima and minima of the function $f(x) = \cos(\pi x)$ using the second derivative test.

Example: For the function $f(x) = x^6 - x^4$, the second derivative test is inconclusive at $x = 0$. Can you nevertheless see what the nature of the critical point 0 is?

Example: Also for the function $f(x) = x^4$, the second derivative test is inconclusive at $x = 0$. The second derivative is zero. Can you nevertheless see whether the critical point 0 is a local maximum or a local minimum?

Let us look at an example, where we can review the chain rule.

Example: Find the critical points of $f(x) = 4\arctan(x) + x^2$. **Solution**. The derivative is

$$f'(x) = \frac{4}{1+x^2} + 2x = \frac{2x + 2x^3 + 4}{1+x^2} .$$

We see that $x = -1$ is a critical point. There are no other roots of $2x + 2x^3 + 4 = 0$. How did we get the derivative of arctan again? Differentiate:$\tan(\arctan(x)) = x$ and write $u = \arctan(x)$:

$$\frac{1}{\cos^2(u)} \arctan'(x) = 1 .$$

Use the identity $1 + \tan^2(u) = 1/\cos^2(u)$ to write this as

$$(1 + \tan^2(u)) \arctan'(x) = 1 .$$

But $\tan(u) = \tan(\arctan(x)) = x$ so that $\tan^2(u) = x^2$. And we have $(1+x^2)\arctan'(x) = 1$. Solving for $\arctan'(x)$ gives $\arctan'(x) = \frac{1}{1+x^2}$.

Example: Finally, let us look at the function $\cosh(x) = (e^x + e^{-x})/2$. We have seen this function already in the last homework. It is the chain curve, because its graph is the shape of a chain. Its sibling is $\sinh(x)$, spelled "sinch" like "grinch".

Homework

Problem 11.1: Find all critical points for the following functions. If there are infinitely many, indicate their structure. For $f(x) = \cos(x)$ for example, the critical points can be written as $\pi/2 + k\pi$, where k is an integer.
a) $f(x) = x^8 - 4x^2$.
b) $f(x) = 7 + 6\sin(2\pi x) + 3$
c) $f(x) = 3\exp(-x^2)x^2$.
d) $f(x) = 2\sin(\cos(\pi x))$

Problem 11.2: Find all the critical points and use the second derivative test to determine whether they are maxima or minima.
a) $f(x) = 2/(1 + x^2)$
b) $f(x) = x^2 - 2x + 1$.
c) $f(x) = 4x\tan(x)$, where $-\pi/2 < x < \pi/2$ d) $f(x) = -x\log(x)$, where $x > 0$.

Problem 11.3: a) Verify that any cubic equation $f(x) = x^3+ax^2+bx+c$ always has an inflection point, a point where $f''(x)$ changes sign.
b) Where is the inflection point for $a = 2, b = 1, c = 3$? Is this point also a critical point? Is it a maximum or minimum?

Problem 11.4: Depending on c, the function $f(x) = x^4 - cx^2$ has either one or three critical points. Use the second derivative test to decide: a) For $c = 1$, find and determine the nature of the critical points.
b) For $c = -1$, find and determine the nature of the critical points.

Problem 11.5: a) Find your own function which has exactly one local maximum and local minimum on the real line.
b) Engineer a concrete function which has exactly 2 local maxima and 1 local minimum.
c) Find a not necessarily continuous function on the real line with 2 local maxima and no local minimum. If it works, give one. If it does not work, give a reason why it does not work.
d) Is there a differentiable function for which we have 2 local maxima and no local minimum? If it works, give one. If it does not work, give a reason why it does not work.