CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 31: Systems of Diff equations

Systems

31.1. Assume we are interested in two linked quantities $x(t), y(t)$ coupled by a **system of differential equations**

$$\begin{aligned} x'(t) &= f(x,y) \\ y'(t) &= g(x,y) \ . \end{aligned}$$

Given an initial condition $(x(0), y(0))$, we can look at the **trajectory** $(x(t), y(t))$ in the plane. What happens in the long term?

31.2. We know already the case

$$\begin{aligned} x'(t) &= y \\ y'(t) &= F(x) \ . \end{aligned}$$

This can be combined to the second order system $x''(t) = F(x)$. For $F(x) = -x$ for example, we get $x'' = -x$ which is the Harmonic oscillator $x'' + x = 0$ we have studied earlier. We study now its solution in the $x-y$ plane.

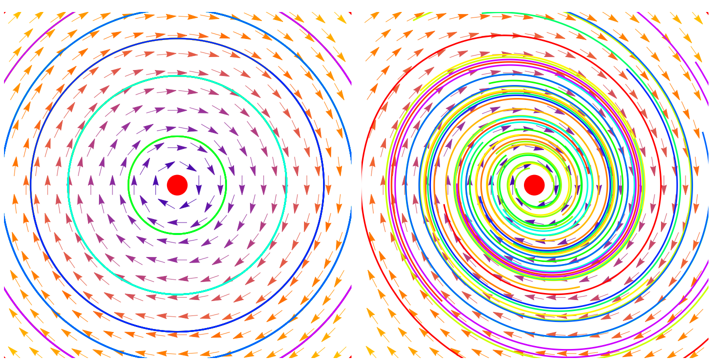

FIGURE 1. The harmonic oscillator without and with friction.

31.3. An other example is an **uncoupled system**

$$\begin{aligned} x'(t) &= f(x) \\ y'(t) &= g(y) \ . \end{aligned}$$

In that case we can look at the two systems $x = f(x)$ and $y' = g(y)$ separately. Also here, it is helpful already to look what happens in the xy-plane.

31.4. A coupled logistic system without interaction:

$$\begin{aligned} x'(t) &= x(6-2x) \\ y'(t) &= y(4-y) \end{aligned}$$

We have looked at such systems and seen that the non-zero equilibrium is stable. We therefore know $x(t) \to 3$ and $y(t) \to 4$.

31.5. Here is the same system where two species are **competing with each other**. If y gets larger, it hurts the growth rate of x. If x gets larger, it hurts the growth rate of y:

$$\begin{aligned} x'(t) &= x(6-2x) - xy \\ y'(t) &= y(4-y) - xy \end{aligned}$$

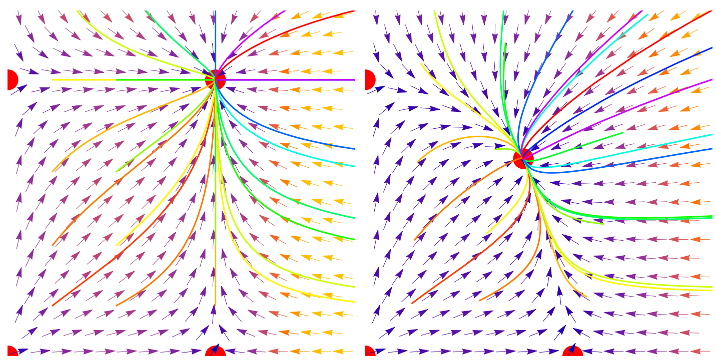

FIGURE 2. A competing population model without or with competition.

31.6. And here is the **slope field story**

$$\begin{aligned} x'(t) &= 1 \\ y'(t) &= f(x) \end{aligned}$$

This means $x(t) = t$ and so $y'(t) = f(t)$. We finally see the reason for using $y(t)$ there.

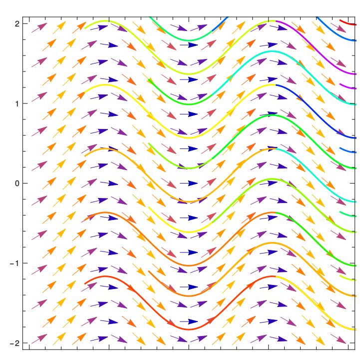

FIGURE 3. A slope field situation for the system $y'(t) = \sin(3t)$.

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