Lecture 25: Differential equations

Differential equations

25.1. A **differential equation** is an equation for an unknown function y involving derivatives of the function. For example, $\boxed{y'(t) = y(t)}$ is a differential equation for an unknown function $y(t)$. We often think of t as "time".

25.2. Unlike for usual equations like $3x = 4$, where we look for a **number** as a solution, we now look for a **function**. A **solution** to the differential equation is a function $y(t)$ which satisfies the equation. You might notice that there is more than one solution to the equation $y'(t) = y(t)$. Can you see the general solution?

25.3. We are already dealing with differential equations when integrating: the equation $\boxed{y'(t) = t^2}$ has the solution $y(t) = \int_0^t x^2 \, dx + C$, where C is a constant. A **specific solution** to the above equation $y'(t) = t^2$ is $t^3/3$. The **general solution** is $t^3/3 + C$.

Growth models

25.4. Here is a typical **population growth** problem:

> **Example:** If $M(t)$ is the number of Canadian geese on the Charles river. Each geese has 2 off-springs a year in average, while 1 of the geese dies. How many geese are there in 5 years, if there are 5000 geese initially?

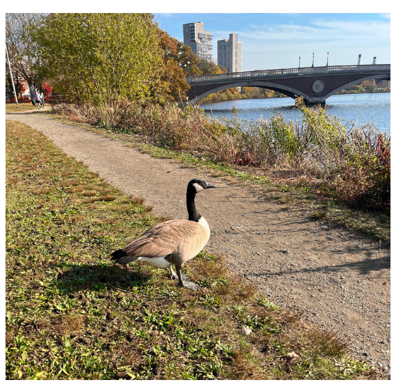

Figure 1. A proud Harvard Canadian goose. Photo shot by Oliver on November 7, 2023.

We can model this as $M'(t) = 2M(t) - M(t) = M(t)$. The solution of $M(t) = 5000e^t$ gives for $t = 5$ the number $5000e^5$.

Decay models

25.5. An other situation, where differential equations appear are **decay models**. Here is a typical example.

> **Example:** The amount $N(t)$ of Carbon 14 in a sample satisfies
> $$N'(t) = -0.0001216N(t)$$

The negative sign means that the number of Carbon 14 isotopes **decreases** in time. If we have initially N_0 atoms, then after time t, we have $M(t) = e^{-0.0001216t}M(0)$. Since the decay number is $\log(2)/5700 = -0.0001216$ we know that in 5700 years, the amount of Carbon 14 is half. We could also write $M(t) = e^{-t/5700}M(0)$.

Banking

25.6.

> **Example:** If $M(t)$ is the bank account, then under a continuous compounding assumption, the balance $M'(t)$ is $rM(t)$, where $r = 0.06$ is the **interest rate**. If there are $M(0) = 100'000$ dollars initially, the equation how much do we have in $t = 10$ years?

The equation $\boxed{M' = rM}$ is solved by $M(t) = 100'000e^{0.06t}$.

> **Example:** If additionally, $50'000$ dollars is continuously transferred to the account we can model this with $\boxed{M'(t) = M(t) + 50'000}$. Can you see why $M(t) = 100'000e^{0.06t} + 50000(e^{0.06t} - 1)$ satisfies the differential equation?

Solution: subtract $M(t)$ from
$$M'(t) = 0.06 * 100'000e^{0.06t} + 0.06 * 50'000e^{0.06t}$$
This gives $50'000$.

25.7. Compare that if r is the interest rate and we would compound annually in n steps, the bank account grows like
$$(1 + \frac{r}{n})^n \ .$$
This is a compound interest formula and for $n \to \infty$ it approaches the function e^r.

25.8. Differential equations can be much more general and model much more interesting situations. Coming up with a differential equation which captures the situation means **making a model**. It is not easy to come up with good models. The British statistician **George Box** once expressed this in 1978 with the following aphorism: *"All models are wrong but some are useful"*.