CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 23: Remainder Theorem

Convergence

23.1. The Taylor approximation of a function f at a point c is the polynomial

$$P_n(x) = \sum_{k=0}^{n} f^{(k)}(c)\frac{(x-c)^k}{k!} .$$

We say it **converges** at x if $P_n(x) \to f(x)$. In that case we have

$$f(x) = \sum_{k=0}^{\infty} f^{(k)}(c)\frac{(x-c)^k}{k!} .$$

In order to justify that we indeed have f we need to estimate the difference between f and P_n.

The error term

Definition: The **error term** is defined as $|f(x) - P_n(x)|$. Given $x = c + d$ with $d < R$, define $M_{(n)}$ as the maximum of $|f^{(n)}(t)|$ for t between $c - d$ and $c + d$.

The **Remainder theorem** tells:

For every x on $[c-d, c+d]$ the error term satisfies the bound

$$|f(x) - P_n(x)| \leq M_{(n+1)}\frac{(x-c)^{n+1}}{(n+1)!}$$

Example: For $f(x) = e^x$ and $c = 0$ we have

$$R_n(x) = e^x\frac{(x-0)^{n+1}}{(n+1)!} .$$

For $x = 1$, this gives the bound $R_n(1) = e/(n+1)!$ which for $n = 5$is $R_5(1) = e/720 = 0.00377$. The actual error is $e - (1 + 1 + 1/2 + 1/6 + 1/24 + 1/120) = 0.00161$.

23.2. The key for verifying the Lagrange error formula is a **single magic equation** for functions f which can be differentiated $n+1$ times continuously, in the sense that $f^{(n+1)}(x)$ is continuous in the region we are interested in. This **magic formula** holds perfectly for all such functions:

$$f(x) = \sum_{k=0}^{n} f^{(k)}(c)\frac{(x-c)^k}{k!} + \int_c^x (x-s)^n f^{(n+1)}(s)\, ds \frac{1}{n!}$$

This is a mouthful! And we just have survived Halloween. But it is beautiful because it is not just an estimate. It is exact! Lets try to understand this:

23.3. For $n = 0$, this means $f(x) = f(c) + \int_c^x f'(s)\, ds$. The magic formula boils down to the **fundamental theorem of calculus**!

23.4. To see this in general, apply **integration by parts** to the expression

$$I_n = \int_c^x (x-s)^n f^{(n+1)}(s)\, ds \frac{1}{n!}$$

Use $(x-s)^n = v$ and $f^{(n+1)}(s) = du$. We see this gives $uv - \int u dv = f^{(n)}(c)(x-c)^n/n! - \int_c^x n(x-s)^{n-1} f^{(n)}(s)\, ds/n! = f^{(n)}(c)(x-c)^n/n! - I_{n-1}$. We have just seen that $I_n - I_{n-1} = f^{(n)}(c)(x-c)^n/n!$ which is just the n'the term in the Taylor expansion.

23.5. Now, lets see why the magic formula implies the Lagrange error claim. The part $\int_c^x (x-s)^n f^{(n+1)}(s)\, ds \frac{1}{n!}$ can in absolute value be estimated by

$$M_{(n+1)} \frac{(x-c)^{n+1}}{(n+1)!} .$$

23.6. Here is how we can imagine the tale on which the magic formula was first conceived. Of course the picture was created with the help of an AI. We can not draw as well. AI is the **new alchemy**. **We do no more generate gold from base metals** like Newton who was an alchemist, but **we generate intelligence from data**. You can quote me on that!

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