CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 19: Alternating Series

Alternating Series

19.1. A series $\sum_k a_k$ is called **alternating series** if a_k and a_{k+1} have opposite signs for all k.

19.2. The series $\sum_{k=1}^{\infty}(-1)^{k+1}/k$ is alternating. The series $\sum_{k=1}^{\infty}\sin(k)/k$ is not alternating. The series $\sum_k \cos(k\pi)/\ln(k)$ is alternating. The **Grandi series** $1-1+1-1+1-\cdots$ is alternating.

Alternating Series test

19.3. Here is the key point of this lecture:

If a_k is alternating and if $|a_k|$ decreases monotonically to zero, then $\sum_k a_k$ converges.

The alternating series test is also called the **Leibniz criterion**.

Error estimate

19.4. As when computing integrals with Riemann sums or when computing partial sums of Taylor series, it is important to know how good a finite sum approximates the result.

If a_k is alternating and $|a_k|$ is monotonically converging to zero, then the partial sum $S_n = a_1 + a_2 + \cdots + a_n$ satisfies $|S - S_n| \leq |a_{n+1}|$.

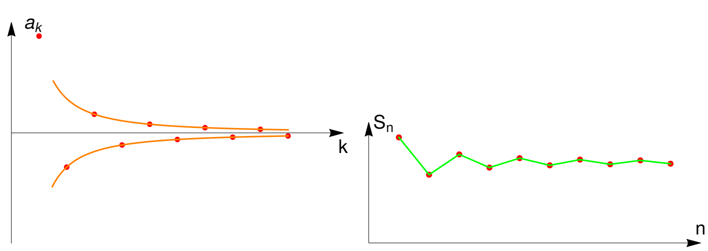

FIGURE 1. A picture explains this. A visualization of the terms a_k and sum S_n for an alternating series.

Examples

19.5. Many Taylor series are alternating. We have seen the Taylor series of $\sin(x)$ or $\cos(x)$ for example.

Example: The Taylor series of $\exp(x)$ at $c = 0$ is

$$\sum_{k=0}^{\infty} \frac{1}{k!} x^k .$$

For negative x, this is an alternating series.

Example: The **Taylor series** for $\ln(1+x)$ is alternating for positive x. The Taylor series is

$$\sum_{k=1}^{\infty} x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$$

For $x = 1$, it is the sum

$$\sum_{k=1}^{\infty} 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$$

It sums up to $\ln(1+1) = \ln(2)$.

Example: The **Leibniz series**

$$\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2k+1}$$

is alternating. Its limit is $\pi/4$.

Remarks

19.6. An alternating series is not necessarily absolutely convergent. The absolute value of the convergent series $\sum_k (-1)^k/k$ is the Harmonic series which diverges. We need that the sequence is alternating. If the series is not absolutely convergent but convergent, it is called **conditionally convergent**.

19.7. Monotonically going to zero means $a_{k+1} \leq a_k$. In most cases we have $a_{k+1} < a_k$. But the theorem does not depend on this we can also have $\leq$.

19.8. We can not skip the assumption of monotonically. We can find sequences a_k which are alternating and go to zero but for which the sum does not converge. For example, take $a_{2k} = 1/k$ and $a_{2k+1} = -1/e^k$. We can then write the sum as two sums. The sum of the even parts is the Harmonic series which diverges. The sum of the odd parts is a convergent series. The total sum diverges.

19.9. We can not skip the assumption that $a_k \to 0$. Take for example the series $\sum_k (1 + 1/k)$. This is a monotonically decreasing series but it does not converge by the **n-th term test**.

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