CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 13: Taylor Series

Taylor series

13.1. We have looked at the **partial sums**

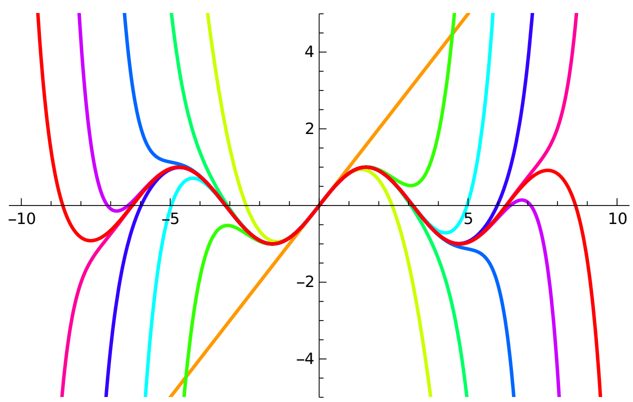
$$P_n(x) = \sum_{k=0}^{n} f^{(k)}(c)\frac{(x-c)^k}{k!} = f(c)+f'(c)\frac{x-c}{1}+f''(c)\frac{(x-c)^2}{2!}+\cdots+f^{(n)}(c)\frac{(x-c)^n}{n!} .$$

If we replace n with ∞, the partial sum is called a **series**.

Definition: The **Taylor series** of a function f at a point c is the series

$$f(x) = \sum_{k=0}^{\infty} f^{(k)}(c)\frac{(x-c)^k}{k!} = f(c) + f'(c)\frac{x-c}{1} + f''(c)\frac{(x-c)^2}{2!} + \ldots .$$

We will talk about **convergence** of these series next time. We will experiment a bit today and see for which this makes sense. The next figure shows the first 10 functions P_1, P_2, P_3 approximating the sin-function. Will this converge if we add infinitely many terms. We will explore this in the next weeks.



Example: For $f(x) = \exp(x)$ we have the series

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

Example: For $f(x) = \exp(x^2)$, just substitute x^2 in the above to get

$$f(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + ...$$

Example: For $f(x) = \sin(x)$ we have the series

$$f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$$

Example: For $f(x) = \sin(x)/x$ (just divide all terms by x, we have

$$f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + ...$$

Example: For $f(x) = \cos(x)$ we have the series

$$f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$$

Example: For $f(x) = 1/(1-x)$ we have the derivatives $f'(x) = -1/(1-x)^2, f''(x) = 2/(1-x)^3, f'''(x) = 6/(1-x)^4, ...$ and so $f^{(k)}(0) = k!$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + ...$$

Example: For $f(x) = \log(1+x))$ we have the series

$$f(x) = \sum_{k=1}^{\infty} (-1)^{(k-1)} \frac{x^k}{k} = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$$

Example: For $f(x) = \arctan(x))$ we have the series

$$f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$$

Verify this by differentiating both sides of $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 \cdots$. Cool application:

$$\pi/4 = \arctan(1) = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$$